Name: Solutions

Directions: Show all work. No credit for answers without work.

1. A traffic study examines the average daily usage of a stretch of road. The study finds that, in the absence of any congestion, the daily usage would increase at a rate of 60 vehicles per day. The effect of congestion is to reduce the daily usage at a rate proportional to the current daily usage, with proportionality constant $0.004(\text{days})^{-1}$. Let y be the daily usage of the road (in ~~numbers of~~ vehicles) at time t (in days).

   (a) **[1 point]** Write a differential equation for y.

$$\frac{dy}{dt} = 60 - 0.004y \quad \Big| \quad \frac{dy}{dt} = 60 - \frac{1}{250}y$$

   (b) **[2 points]** Solve the initial value problem with $y(0) = y_0$.

$$\frac{dy}{dt} = -\frac{1}{250}(y - 15{,}000)$$

$$\frac{1}{y-15{,}000}\frac{dy}{dt} = -\frac{1}{250}$$

$$\int \frac{1}{y-15000}\,dy = \int -\frac{1}{250}\,dt$$

$$\ln(y-15{,}000) = -\frac{1}{250}t + C$$

$$y - 15000 = Ce^{-\frac{1}{250}t}$$

$$y_0 - 15000 = C(1)$$

$$\boxed{y = 15{,}000 + (y_0 - 15{,}000)e^{-\frac{1}{250}t}}$$

   (c) **[2 points]** If the average daily usage is currently 700 vehicles, how long will it take for the usage to increase to 90% of the limiting value?

Limiting value: 15,000

$$\frac{9}{10}\cdot 15\cdot 10^3 = 15\cdot 10^3 + (7\cdot 10^2 - 15\cdot 10^3)e^{-\frac{1}{250}t}$$

$$(150-7)\cancel{10^2}\,e^{-\frac{1}{250}t} = \frac{1}{10}\cdot 15\cdot \cancel{10^{3}}$$

$$e^{-\frac{1}{250}t} = \frac{15}{143}$$

$$-\frac{1}{250}t = \ln\left(\frac{15}{143}\right)$$

$$t = 250\ln\left(\frac{143}{15}\right)$$

$$\approx \boxed{563.7 \text{ days}}$$

2. **[2 points]** Determine the values of r for which $w = e^{rt}$ is a solution to $\frac{d^2w}{dt^2} + 3\frac{dw}{dt} - 4w = 0$.

$w' = re^{rt}$

$w'' = r^2e^{rt}$

$w'' + 3w' - 4w = 0$

$r^2e^{rt} + 3re^{rt} - 4e^{rt} = 0$

$e^{rt}(r^2 + 3r - 4) = 0$

$e^{rt} = 0$ or $r^2 + 3r - 4 = 0$

No soln $\quad (r+4)(r-1) = 0$

$\boxed{r = -4 \text{ or } r = 1}$

3. **[3 points]** Solve the initial value problem $y' + \frac{3}{t}y = \frac{\cos t}{t^2}$ with $y(\pi) = 1$ and $t > 0$.

$\mu = e^{\int \frac{3}{t}dt} = e^{3\ln(t)} = e^{\ln(t^3)} = t^3$

$t^3y' + 3t^2y = t\cos t$

$\frac{d}{dt}[t^3y] = t\cos t$

$t^3y = \int t\cos t\,dt \qquad u = t \quad v = \sin t \qquad du = dt \quad dv = \cos t\,dt$

$t^3y = t\sin t - \int \sin t\,dt$

$t^3y = t\sin t + \cos t + C$

$\underline{y(\pi) = 1}$

$\pi^3 \cdot 1 = \pi \overset{0}{\cancel{\sin(\pi)}} + \overset{-1}{\cancel{\cos(\pi)}} + C$

$C = \pi^3 + 1$

$\boxed{y = \frac{1}{t^3}\left[t\sin t + \cos t + \pi^3 + 1\right]}$