Math545 **Homework 5** **Due Wed, Nov 13, 2013**

Directions: Solve the following 6 problems. See the course syllabus and the Homework Webpage on the course website for general directions and guidelines.

1. [NT 8-1.16] Let $n = 132!$. How many zeros are at the end of the base 2 representation of n? How many zeros are at the end of the base 2 representation of n?

2. [NT 9-1.1(c)] Use Euler's Criterion to determine whether 3 is a quadratic residue modulo 11. Note: use Euler's Criterion, not quadratic reciprocity or other techniques.

3. [NT 9-3.5] Use the Quadratic Reciprocity Law to prove that

$$\left(\frac{3}{p}\right) = \begin{cases} 1 & \text{if } p \equiv 1 \text{ or } 11 \pmod{12} \\ -1 & \text{if } p \equiv 5 \text{ or } 7 \pmod{12} \end{cases}$$

   for each prime p where $p \geq 5$.

4. [NT 9-4] Determine (with proof) whether the following congruences have solutions.
   (a) $x^2 \equiv 17 \pmod{29}$
   (b) $3x^2 \equiv 12 \pmod{23}$
   (c) $2x^2 \equiv 27 \pmod{41}$
   (d) $x^2 + 5x \equiv 12 \pmod{31}$ *Hint:* complete the square.
   (e) $x^2 \equiv 19 \pmod{30}$

5. Legendre sums. For an odd prime p, let $T(p) = \sum_{a \leq (p-1)/2} \left(\frac{a}{p}\right)$.
   (a) Prove that $T(p) = 0$ if and only if $p \equiv 1 \pmod 4$.
   (b) Use sage to find $\max_{p \leq 20{,}000} |T(p)|$. Which prime maximizes $T(p)$? *Sage Hint:* the function kronecker(a, p) returns the value of the Legendre symbol $\left(\frac{a}{p}\right)$. Note: no code overview required for this problem.

   Remark: I suspect that $|T(p)|$ is much smaller than p, perhaps it is $O(\sqrt{p})$.

6. A prime p is *lonely* in a if $p \mid a$ but $p^2 \nmid a$. A number a is *special* if it has no lonely primes.
   (a) Find a pair of consecutive special numbers.
   (b) Prove that there are infinitely many pairs of consecutive special numbers. *Hint*: given a pair $\{a, a+1\}$ of special numbers, construct another pair $\{b, b+1\}$ of special numbers with $b > a$.
   (c) Execute your proof to obtain two more pairs of consecutive special numbers. Note: the purpose of this part is for you to check your proof in part (b), so we do not want just any old pairs. We want the ones your proof produces.
   (d) Prove that there are no consecutive blocks of special numbers of size 4. That is, prove that at least one integer in $\{a, a+1, a+2, a+3\}$ is not special.

   Remark: this naturally begs the question: are there consecutive blocks of special numbers of size 3? I do not know the answer to this question, but numerical evidence from sage suggests the answer may be no.