**Directions:** Solve the following 6 problems. For computer problems, complete solutions include source code, an **English language overview** of your code, and the final answer. The overview should be sufficiently complete that an intelligent reader with no prior access to your code will understand how it works. See the course syllabus and the Homework Webpage on the course website for general directions and guidelines.

1. [NT 2-2.4] The *least common multiple* of $a$ and $b$, denoted $\text{lcm}(a, b)$, is the smallest positive integer $\ell$ such that $a \mid \ell$ and $b \mid \ell$. Prove that if $a$ and $b$ are positive integers, then $\text{lcm}(a, b) = ab/\gcd(a, b)$.

2. [NT 2-3.1] Find the general solution (if solutions exist) of each of the following linear Diophantine equations:

   (a) $15x + 51y = 41$

   (b) $23x + 29y = 25$

   (c) $121x - 88y = 572$

3. Binomial Coefficients and Parity I.

   (a) [NT 3-1.3] Using the definition of $\binom{n}{r}$, show combinatorially that $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$.

   (b) Prove that if $n$ is even and $r$ is odd, then $\binom{n}{r}$ is even.

4. Binomial Coefficients and Parity II. Pascal's Triangle is an arrangement of the binomial coefficients in which $\binom{n}{k}$ is placed at position $k$ in row $n$. For example, the rows $n = 0$ through $n = 6$ of Pascal's Triangle are shown on the left:

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}$$

$$\begin{array}{ccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 2 & 3 & 4 & 5 & 6 & \\
1 & 3 & 6 & 10 & 15 & & \\
1 & 4 & 10 & 20 & & & \\
1 & 5 & 15 & & & & \\
1 & 6 & & & & & \\
1 & & & & & &
\end{array}$$

   For the computer problem, it is convenient to rotate Pascal's Triangle so that $\binom{x+y}{x}$ appears in position $(x, y)$, as shown to the right. Write a program whose output is a $(30 \times 30)$-array of characters with an asterisk in position $(x, y)$ if $\binom{x+y}{x}$ is even and a blank space otherwise. Turn in your an source code and a picture of the grid. (As an exception for this programming problem, no English description is required.)

5. Binomial Coefficients and Parity III. A positive integer $n$ is *excellent* if the set $\left\{\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}\right\}$ contains only odd integers.

   (a) Which numbers are excellent? Based on examining data, formulate a hypothesis.

   (b) **[Challenge]** Prove that your hypothesis is correct.

6. [NT 3-2.3] Prove that $n^5$ and $n$ have the same last digit.