1. [EC 13.2.34] Find the work done by the force field $\vec{F}(x, y) = x \sin y\vec{i} + y\vec{j}$ on a particle that moves along the parabola $y = x^2$ from $(-1, 1)$ to $(2, 4)$.

2. [EC 13.3.{4,6,8}] Determine whether or not $\vec{F}$ is a conservative vector field. If it is, find a function f such that $\vec{F} = \nabla f$.

   (a) $\vec{F}(x, y) = (x^3 + 4xy)\vec{i} + (4xy - y^3)\vec{j}$

   (b) $\vec{F}(x, y) = e^y\vec{i} + xe^y\vec{j}$

   (c) $\vec{F}(x, y) = (1 + 2xy + \ln x)\vec{i} + x^2\vec{j}$

3. [EC 13.3.14] Find a function f such that $\nabla f = \vec{F}$ and use it to evaluate $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F}(x, y, z) = (2xz + y^2)\vec{i} + 2xy\vec{j} + (x^2 + 3z^2)\vec{k}$ and C is the curve given by $\vec{r}(t) = t^2\vec{i} + (t + 1)\vec{j} + (2t - 1)\vec{k}$ for $0 \leq t \leq 1$.

4. [EC 13.4.2] Evaluate the line integral first directly, and then using Green's Theorem: $\int_C y\,dx - x\,dy$, where C is the unit circle centered at the origin.

5. [EC 13.4.8] Use Green's Theorem to evaluate the line integral $\int_C x^2y^2\,dx + 4xy^3\,dy$ where C is the positively oriented curve along the triangle with vertices $(0, 0)$, $(1, 3)$, and $(0, 3)$.

Solutions

1. • Parameterize curve C from $(-1, 1)$ to $(2,4)$ along $y=x^2$:

$\vec{r}(t) = \langle t, t^2 \rangle$, for $-1 \le t \le 2$. (i.e. $x=t$, $y=t^2$)

• $\vec{r}\,'(t) = \langle 1, 2t \rangle$

• $W = \int_C \vec{F} \cdot d\vec{r} = \cancel{\int_{-1}^{2} \langle x\sin y, y \rangle \cdot \langle n}$

$$= \int_C \vec{F} \cdot \vec{r}\,' \, dt$$

$$= \int_{-1}^{2} \langle x\sin y, y \rangle \cdot \langle 1, 2t \rangle \, dt$$

$$= \int_{-1}^{2} \langle t\sin t^2, t^2 \rangle \cdot \langle 1, 2t \rangle \, dt$$

$$= \int_{-1}^{2} t\sin t^2 + (t^2)(2t) \, dt$$

$$= \int_{-1}^{2} t\sin(t^2) + 2t^3 \, dt$$

$$= \int_{-1}^{2} t\sin(t^2) \, dt + \left. \frac{t^4}{4} \right|_{t=-1}^{t=2}$$

$u = t^2$, $du = 2t\,dt$

$$= \int_{1}^{4} \sin(u) \frac{du}{2} + \left(\frac{2^4}{4} - \frac{(-1)^4}{4} \right)$$

$$= \cancel{\int_1} \; \frac{-\cos(u)}{2}\Bigg|_{u=1}^{u=4} + \left(8 - \frac{1}{2}\right)$$

$$= \left(\frac{-\cos(4)}{2}\right) - \left(\frac{-\cos(1)}{2}\right) + \frac{15}{2}$$

$$= \boxed{\frac{1}{2}\left(\cos(1) - \cos(4) + 15\right)}$$

2a. $\dfrac{\partial P}{\partial y} = \dfrac{\partial}{\partial y}\left[x^3 + 4xy\right]$

$= 4x$

$\dfrac{\partial Q}{\partial x} = \dfrac{\partial}{\partial x}\left[4xy - y^3\right] = 4y$

not equal, so $\vec{F}$ is not conservative.

b. $\dfrac{\partial P}{\partial y} = e^y$, $\dfrac{\partial Q}{\partial x} = e^y$, so conservative.

- Find f such that $f_x = e^y$, $f_y = xe^y$:

$f(x,y) = \int f_x \, dx = \int e^y \, dx = xe^y + g(y)$

- $\cancel{xe^y} = f_y = \cancel{xe^y} + g'(y)$, $g'(y) = 0$, so $g(y) = \int 0 \, dy = C$

Choosing $C = 0$: $\boxed{f(x,y) = xe^y}$

C. * Note Typo in original worksheet *

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left[1 + 2xy + \ln x\right] = 2y$$

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\left[x^2\right] = 2x$$

not equal, so F is not conservative.

3. • For fixed y, z:

$$f(x,y,z) = \int f_x\,dx = \int 2xz + y^2\,dx = zx^2 + y^2x + g(y,z)$$

• Want $f_y = 2xy$:

$$\cancel{2xy} = f_y = \frac{\partial}{\partial y}\left[zx^2 + y^2x + g(y,z)\right]$$

$$= 0 + \cancel{2yx} + g_y(y,z)$$

• $g(y,z) = \int g_y(y,z)\,dy$ for fixed z

$$= \int 0\,dy = 0 + h(z)$$

• So, $f(x,y,z) = zx^2 + y^2x + h(z)$.

- Want $f_z = x^2 + 3z^2$:

$$\cancel{x^2} + 3z^2 = f_z = \frac{\partial}{\partial z}\left[zx^2 + y^2x + h(z)\right]$$

$$= \cancel{x^2} + 0 + h'(z)$$

- So $h'(z) = 3z^2$. Therefore

$$h(z) = \int h'(z)\,dz = \int 3z^2\,dz = z^3 + C;$$

- with $C=0$:

$$\boxed{f(x,y,z) = zx^2 + y^2x + z^3}$$

- By FTC for Line Integrals:

$$\int_C \vec{F} \cdot d\vec{r} = f(\vec{r}(1)) - f(\vec{r}(0))$$

- $\vec{r}(1) = 1^2\vec{i} + 2\vec{j} + \vec{k} = \langle 1, 2, 1\rangle$.
- $\vec{r}(0) = 0^2\vec{i} + 1\vec{j} - \vec{k} = \langle 0, 1, -1\rangle$.

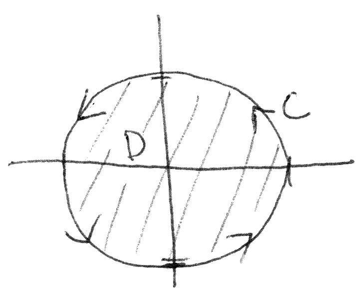
So, $\int_C \vec{F}\cdot d\vec{r} = f(\vec{r}(1)) - f(\vec{r}(0))$

$$= f(\overset{x}{1},\overset{y}{2},\overset{z}{1}) - f(\overset{x}{0},\overset{y}{1},\overset{z}{-1})$$

$$= (1\cdot 1^2 + 2^2\cdot 1 + 1^3) - ((-1)\cdot 0^2 + 1^2\cdot 0 + (-1)^3)$$

$$= (1+4+1) - (-1) = \boxed{7}.$$

4.



$C: \vec{r}(t) = \underbrace{\cos t}_{x(t)}\,\vec{i} + \underbrace{\sin t}_{y(t)}\,\vec{j}, \quad 0 \le t \le 2\pi.$

Direct Computation:

$$\int_C y\,dx - x\,dy = \int_0^{2\pi} (y x' - x y')\,dt$$

$$= \int_0^{2\pi} \left((\sin t)\frac{d}{dt}[\cos t] - (\cos t)\frac{d}{dt}[\sin t] \right) dt$$

$$= \int_0^{2\pi} \left(-(\sin t)^2 - (\cos t)^2 \right) dt = \int_0^{2\pi} -1\,dt$$

$$= \boxed{-2\pi}.$$

· Via Green's Theorem:

$$\int_C \underbrace{y}_{P(x,y)}\,dx \underbrace{- x}_{Q(x,y)}\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$= \iint_D (-1 - 1)\, dA$$

$$\cancel{= -2 \iint_D dA}$$

$$= -2 \boxed{\iint_D 1\, dA}$$

area of unit circle, $= \pi r^2$ with $r = 1$, OR, we can finish with polar integration:

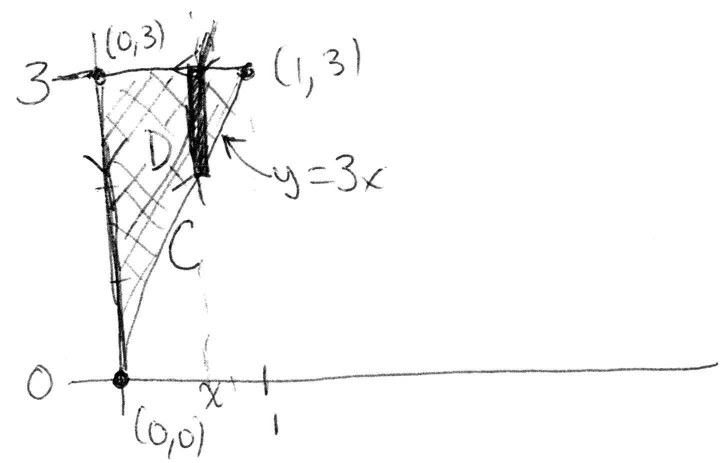
$$= -2 \int_0^{2\pi} \int_0^1 1 \; r\,dr\,d\theta$$

$$= -2 \int_0^{2\pi} \left. \frac{r^2}{2} \right|_0^1 d\theta$$

$$= -2 \int_0^{2\pi} \frac{1}{2}\, d\theta$$

$$= -\left(\theta \Big|_0^{2\pi} \right) = \boxed{-2\pi}$$

5.



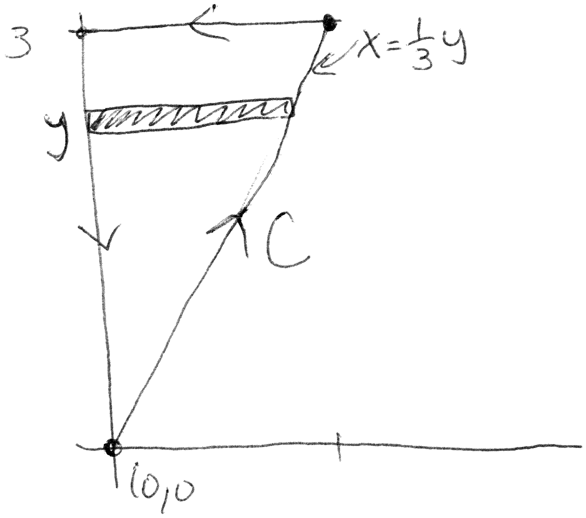
$$\int_C \underbrace{x^2y^2}_{P}\,dx + \underbrace{4xy^3}_{Q}\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \iint_D \left(4y^3 - 2x^2y\right) dA$$

$$= \int_0^1 \left[\int_{3x}^{3} \left(4y^3 - 2x^2y\right) dy\right] dx$$

$$= \int_0^1 \left(y^4 - x^2y^2\right)\Big|_{3x}^{3} dx$$

Hmm. That doesn't look like fun. Let's ~~start over~~ try viewing D as a type II region with horizontal rectangles.



$$\iint_{D} (4y^3 - 2x^2y)\,dA$$

$$= \int_0^3 \left[\int_0^{\frac{1}{3}y} (4y^3 - 2x^2y)\,dx \right] dy$$

moderately nicer → $$= \int_0^3 \left(4y^3x - \frac{2}{3}x^3y\right)\Big|_{x=0}^{x=\frac{1}{3}y} dy$$

$$= \int_0^3 \left(\frac{4}{3}y^4 - \frac{2}{3}\left(\frac{1}{3}y\right)^3 y\right) - (0)\,dy$$

$$= \int_0^3 \frac{4}{3}y^4 - \frac{2}{81}y^4\,dy$$

$$= \cancel{\frac{4}{5}}\int_0^3 \left(\frac{108}{81} - \frac{2}{81}\right)y^4\,dy = \int_0^3 \frac{106}{81}y^4\,dy = \frac{106}{5\cdot 81}y^5\Big|_0^3$$

$$= \frac{106}{5\cdot 81}\left(3^5\right) = \frac{106}{5\cdot 3^4}\cdot 3^5 = \frac{106\cdot 3}{5} = \boxed{\frac{318}{5}}$$