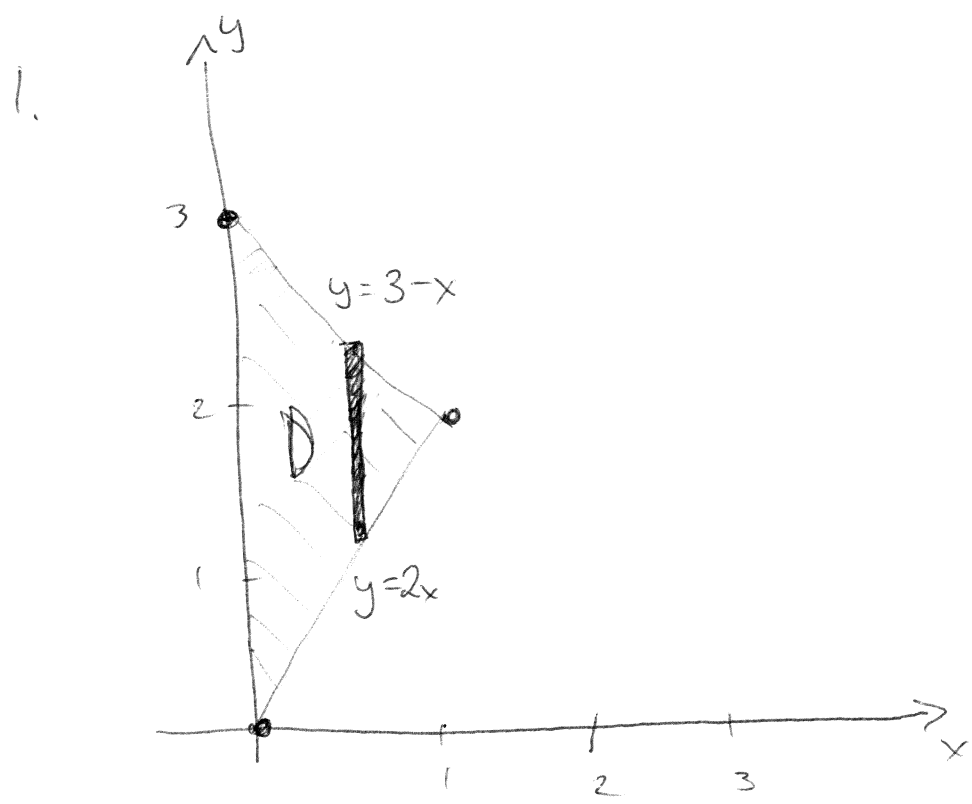


1. [EC 12.2.16] Compute $\iint_D 2xy \, dA$ where D is the triangular region with vertices $(0, 0)$, $(1, 2)$, and $(0, 3)$.
2. [EC 12.2.26] Find the volume of the solid bounded by the cylinders $x^2 + y^2 = r^2$ and $y^2 + z^2 = r^2$.
3. [EC 12.2.{40,42}] Evaluate the following.
 - (a) $\int_0^1 \int_{x^2}^1 x^3 \sin(y^3) \, dy \, dx$
 - (b) $\int_0^8 \int_{y^{1/3}}^2 e^{x^4} \, dx \, dy$
4. [EC 12.3.6] Sketch the region whose area is given by the integral and evaluate the integral:
 $\int_0^{\pi/2} \int_0^{4 \cos \theta} r \, dr \, d\theta$
5. [EC 12.3.18] Use polar coordinates to find the volume of the solid bounded by the paraboloid $z = 1 + 2x^2 + 2y^2$ and the plane $z = 7$ in the first octant (where x , y , and z are all at least 0).

Solutions



$$\iint_D 2xy \, dA = \int_0^1 \left[\int_{2x}^{3-x} 2xy \, dy \right] dx$$

$$= \int_0^1 xy^2 \Big|_{y=2x}^{y=3-x} dx$$

$$= \int_0^1 x(3-x)^2 - x(2x)^2 dx$$

$$= \int_0^1 x(3-x)^2 dx - \int_0^1 4x^3 dx$$

$$u=3-x$$

$$du=-dx$$

$$= \int_3^2 (3-u)u^2(-du) - x^4 \Big|_{x=0}^{x=1}$$

(2)

$$= \int_2^3 3u^2 - u^3 du - 1$$

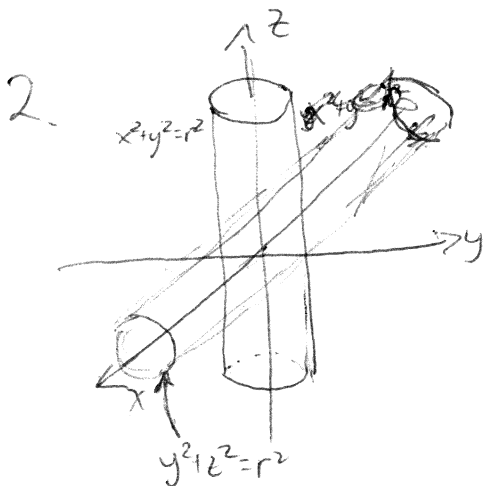
$$= \left(u^3 - \frac{u^4}{4} \right) \Big|_{u=2}^{u=3} - 1$$

$$= \left(3^3 - \frac{3^4}{4} \right) - \left(2^3 - \frac{2^4}{4} \right) - 1$$

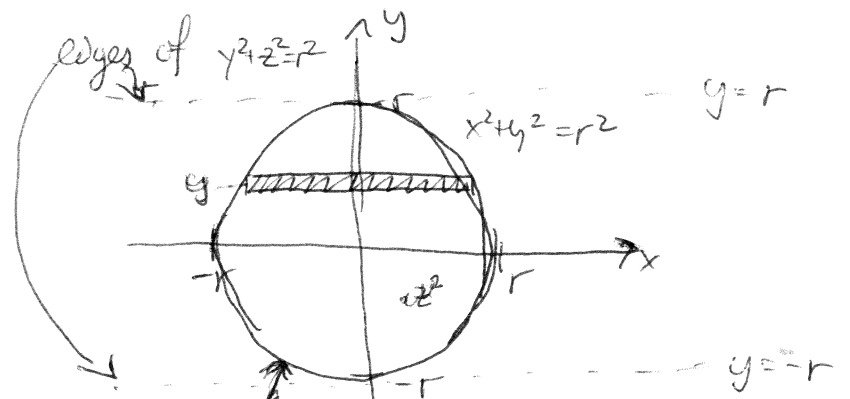
$$= 3^3 \left(1 - \frac{3}{4} \right) - 2^3 \left(1 - \frac{2}{4} \right) - 1$$

$$= 27 \cdot \frac{1}{4} - 8 \cdot \frac{1}{2} - 1$$

$$= \frac{27}{4} - 5 = \boxed{\frac{7}{4}}$$



In $z=0$ plane:



Cylinder orthogonal to page; no vertical restriction from this.

(3)

Using horizontal rectangles,

• Each point (x, y) extends a column above and below

to height z where $y^2 + z^2 = r^2$, so $z^2 = r^2 - y^2$,

$z = \pm \sqrt{r^2 - y^2}$. So at (x, y) , the column has

height $(+\sqrt{r^2 - y^2}) - (-\sqrt{r^2 - y^2}) = 2\sqrt{r^2 - y^2}$.

• This suggests horizontal rectangles, which have constant y

values. For a given y in $[-r, r]$, x ranges from $-\sqrt{r^2 - y^2}$ to $+\sqrt{r^2 - y^2}$

$$\bullet \text{ Volume} = \int_{-r}^r \int_{-\sqrt{r^2 - y^2}}^{\sqrt{r^2 - y^2}} 2\sqrt{r^2 - y^2} \, dx \, dy$$

$$= \int_{-r}^r 2\sqrt{r^2 - y^2} \, x \Big|_{x=-\sqrt{r^2 - y^2}}^{x=\sqrt{r^2 - y^2}} \, dy$$

$$= \int_{-r}^r 2\sqrt{r^2 - y^2} \left((\sqrt{r^2 - y^2}) - (-\sqrt{r^2 - y^2}) \right) \, dy$$

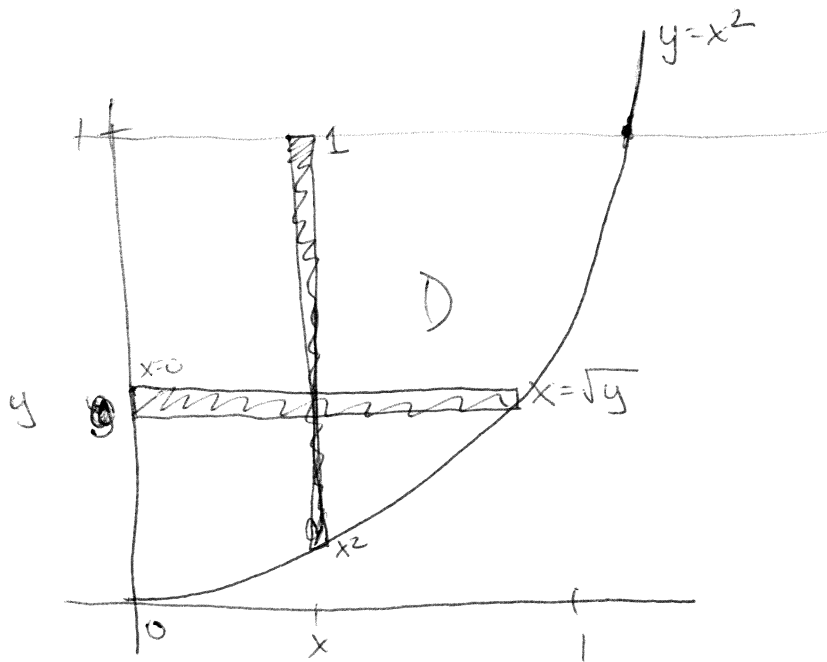
$$= 4 \int_{-r}^r r^2 - y^2 \, dy = 4 \left(r^2 y - \frac{y^3}{3} \right) \Big|_{y=-r}^{y=r}$$

$$= 4 \left(\left(r^3 - \frac{r^3}{3} \right) - \left(-r^3 - \frac{-r^3}{3} \right) \right)$$

(4)

$$= 8\left(\frac{2}{3}r^3\right) = \boxed{\frac{16}{3}r^3}$$

3a



$$\int_0^1 \int_{x^2}^1 x^3 \sin(y^3) dy dx = \iint_D x^3 \sin(y^3) dA$$

$$= \int_0^1 \int_0^{\sqrt{y}} x^3 \sin(y^3) dx dy$$

$$= \int_0^1 \sin(y^3) \cdot \left(\frac{x^4}{4}\right)_{x=0}^{x=\sqrt{y}} dy$$

$$= \int_0^1 \sin(y^3) \cdot \left(\frac{y^2}{4} - 0\right) dy$$

$$= \frac{1}{4} \int_0^1 y^2 \sin(y^3) dy$$

(5)

Set $u=y^3$, $du=3y^2 dy$:

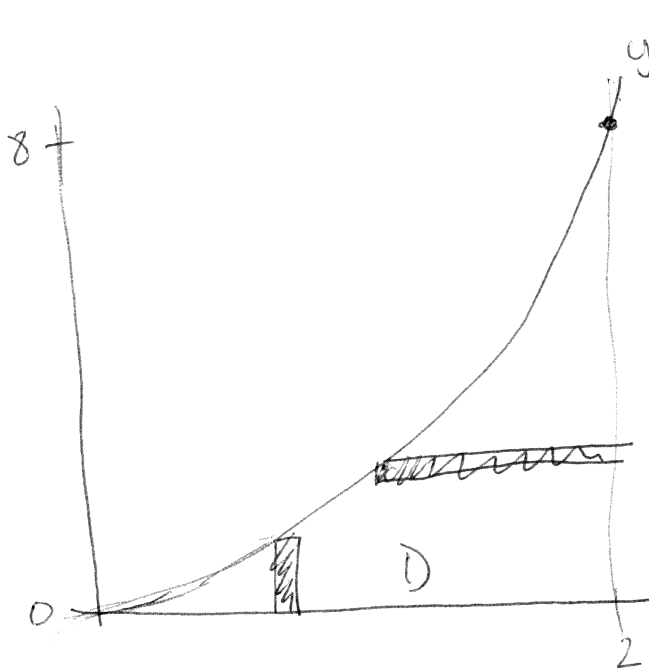
$$= \frac{1}{4} \int_{0^3}^{1^3} \sin(u) \frac{1}{3} du$$

$$= \frac{1}{12} \int_0^1 \sin(u) du$$

$$= \frac{1}{12} (-\cos(u)) \Big|_{u=0}^{u=1} = \frac{1}{12} (-\cos(1) + \cos(0))$$

$$= \boxed{\frac{1 - \cos(1)}{12}}$$

b.



$$y = x^3, x = y^{1/3}$$

Lower limit $x = y^{1/3}$

$$x^3 = y$$

$$\int_0^8 \int_{y^{1/3}}^2 e^{x^4} dx dy$$

$$= \iint_D e^{x^4} dA$$

$$= \int_0^2 \int_0^{x^3} e^{x^4} dy dx$$

⑥

$$= \int_0^2 y e^{x^4} \Big|_{y=0}^{y=x^3} dx$$

$$= \int_0^2 x^3 e^{x^4} dx$$

$$u = x^4, \quad du = 4x^3 dx$$

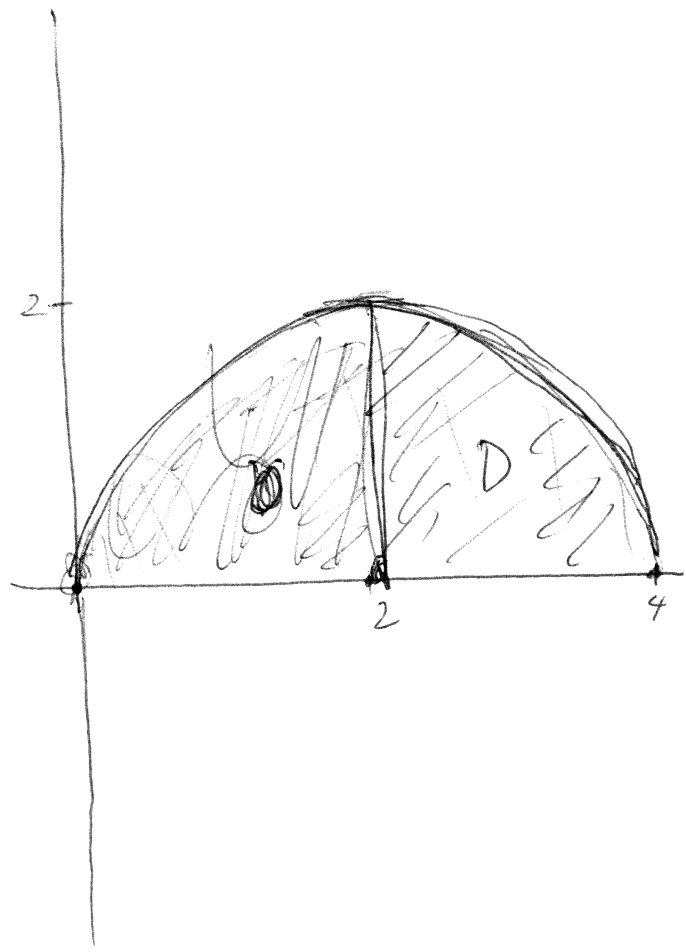
$$= \int_{0^4}^{2^4} e^u \cdot \frac{1}{4} du$$

$$= \frac{1}{4} \int_0^{16} e^u du$$

$$= \frac{1}{4} e^u \Big|_{u=0}^{u=16}$$

$$= \frac{1}{4} (e^{16} - e^0) = \boxed{\frac{e^{16} - 1}{4}}$$

4.



$$r = 4\cos\theta, \quad \theta \text{ in } [0, \pi/2]$$

Note ^{Recall:} $r = a\cos\theta$ as θ ranges from 0 to π traces the circle with center $(\frac{a}{2}, 0)$ and radius $\frac{a}{2}$.

To see why, note:

$$(1) \quad x = r\cos\theta = a\cos^2\theta$$

$$(2) \quad \begin{aligned} x^2 + y^2 &= r^2 \\ x^2 + y^2 &= a^2\cos^2\theta \\ x^2 + y^2 &= a \cdot a\cos^2\theta \\ x^2 + y^2 &= ax \end{aligned}$$

$$\begin{aligned} x^2 - ax + y^2 &= 0 \\ (x - \frac{a}{2})^2 + y^2 &= \frac{a^2}{4} \end{aligned}$$

↑
complete the square

- Our curve is ^{half} part of the circle with center $(2, 0)$ and radius r .

(8)

The full circle has area $\pi \cdot 2^2 = 4\pi$.

- Since D is only half of this circle,

$$\int_0^{\pi/2} \int_0^{4\cos\theta} r dr d\theta = \text{area}(D)$$

$$= \frac{1}{2} \cdot 4\pi = \boxed{2\pi}$$

- Direct Evaluation:

$$\int_0^{\pi/2} \int_0^{4\cos\theta} r dr d\theta = \int_0^{\pi/2} \left. \frac{r^2}{2} \right|_{r=0}^{r=4\cos\theta} d\theta$$

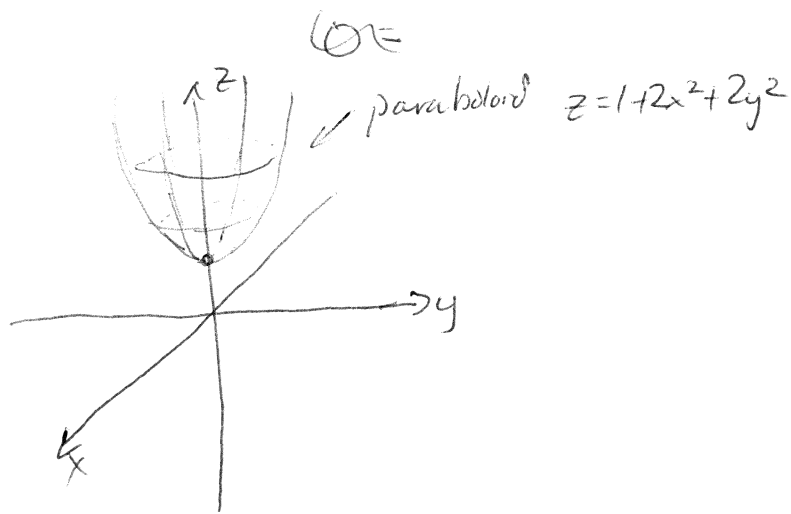
$$= \int_0^{\pi/2} \frac{16\cos^2\theta}{2} - \frac{0^2}{2} d\theta = 8 \int_0^{\pi/2} \cos^2\theta d\theta$$

$$= 8 \int_0^{\pi/2} \frac{1 + \cos(2\theta)}{2} d\theta = 4 \int_0^{\pi/2} (1 + \cos(2\theta)) d\theta$$

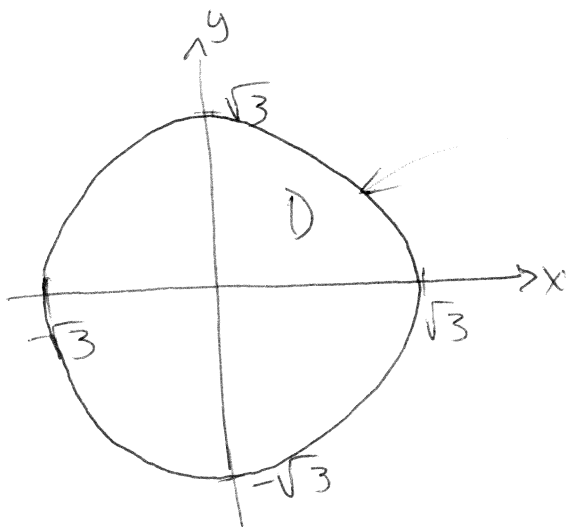
$$= 4 \left(\theta + \frac{\sin(2\theta)}{2} \right) \Big|_{\theta=0}^{\theta=\pi/2} = 4 \left(\left(\frac{\pi}{2} + \frac{\sin(\pi)}{2} \right) - \left(0 + \frac{\sin(0)}{2} \right) \right) \\ = 4 \left(\frac{\pi}{2} + 0 - 0 - 0 \right) = \boxed{2\pi}$$

(9)

5. In plane $z=0$:



• Picture suggests looking at the $z=7$ plane:



$$7 = 1 + 2x^2 + 2y^2$$

$$3 = x^2 + y^2$$

• At (x, y) in D , there is a column extending downward with height Ⓣ

$$7 - z$$

$$\text{or } 7 - (1 + 2x^2 + 2y^2)$$

$$= 6 - 2x^2 - 2y^2.$$

$$\bullet \text{ Volume} = \iint_D 6 - 2x^2 - 2y^2 \, dA$$

$$= \int_0^{2\pi} \int_0^{\sqrt{3}} [6 - 2(r\cos\theta)^2 - 2(r\sin\theta)^2] r \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\sqrt{3}} [6 - 2r^2(\cos^2\theta + \sin^2\theta)] r \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\sqrt{3}} 6r - 2r^3 \, dr \, d\theta$$

$$= \int_0^{2\pi} \left(3r^2 - \frac{1}{2}r^4 \right) \Big|_{r=0}^{r=\sqrt{3}} d\theta$$

$$= \int_0^{2\pi} \left(3(\sqrt{3})^2 - \frac{1}{2}(\sqrt{3})^4 \right) - 0 \, d\theta$$

$$= \int_0^{2\pi} 9 - \frac{9}{2} \, d\theta = \frac{9}{2} \theta \Big|_{\theta=0}^{2\pi} = \frac{9}{2}(2\pi - 0) = \boxed{9\pi}$$