Math251 | **Worksheet 4** | **October 30, 2013**

1. [EC 11.6, evens] Find the directional derivative of the function at the given point in the given direction $\vec{v}$ or with the angle that θ makes with the x-axis.

   (a) $f(x, y) = x\sin(xy)$ at $(2, 0)$ with $\theta = \pi/3$

   (b) $f(x, y, z) = x/(y + z)$, at $(4, 1, 1)$ with $\vec{v} = \langle 1, 2, 3\rangle$.

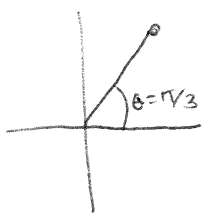
2. [EC 11.6.18] Find the maximum rate of change of $f(x, y, z) = \tan(x + 2y + 3z)$ at $(-5, 1, 1)$ and the direction in which it occurs.

3. [EC 11.6.34] Find the equations of the tangent ~~line~~ plane and normal line to $yz = \ln(x + z)$ at $(0, 0, 1)$.

4. [EC 11.7.8] Find and classify the critical points of $f(x, y) = 2x^3 + xy^2 + 5x^2 + y^2$.

5. [EC 11.7.34] Find the points on the surface $y^2 = 9 + xz$ that are closest to the origin.

1a.



$\vec{u} = \langle \cos\pi/3, \sin\pi/3 \rangle = \langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle$

- $f_x = \sin(xy) + x\cos(xy)\cdot y = \sin(xy) + xy\cos(x,y)$.

  $f_x(2,0) = \sin(0) + 0\cdot\cos(0) = 0$

- $f_y = x\cos(xy)\cdot x = x^2\cos(xy)$.

  $f_y(2,0) = 2^2\cos(0) = 4$

- $D_u f(2,0) = \cancel{f_x(2,0)}\ \nabla f \cdot \vec{u} = \langle 0, 4\rangle \cdot \langle \frac{1}{2}, \frac{\sqrt{3}}{2}\rangle = \boxed{2\sqrt{3}}$

1b. $f_x = \dfrac{1}{y+z}$, $f_y = \dfrac{\partial}{\partial y}\left[x(y+z)^{-1}\right] = -x\cdot\dfrac{1}{(y+z)^2} = \dfrac{-x}{(y+z)^2}$

$f_z = \dfrac{-x}{(y+z)^2}$. So $\nabla f(4,1,1) = \langle \frac{1}{2}, -1, -1\rangle$.

$$D_u f(4,1,1) = \nabla f(4,1,1) \cdot \frac{\vec{v}}{|\vec{v}|}$$

$$= \langle \tfrac{1}{2}, -1, -1 \rangle \cdot \frac{\langle 1, 2, 3 \rangle}{\sqrt{1^2+2^2+3^2}}$$

$$= \frac{1}{\sqrt{14}}\left(\tfrac{1}{2} - 2 - 3\right) = \frac{1}{\sqrt{14}}\left(\tfrac{1}{2} - \tfrac{10}{2}\right) = \boxed{\frac{-9}{2\sqrt{14}}}$$

2. Max rate of change is $|\nabla f|$ in direction of ∇f.

- $f_x = \sec^2(x+2y+3z)\cdot 1$, $f_y = \sec^2(x+2y+3z)\cdot 2$, $f_z = \sec^2(x+2y+3z)\cdot 3$

- $\nabla f(-5,1,1) = \langle \sec^2(0), 2\sec^2(0), 3\sec^2(0) \rangle = \langle 1, 2, 3 \rangle$.

- Max rate of change is $\sqrt{1^2+2^2+3^2} = \boxed{\sqrt{14}}$ in direction of $\boxed{\langle 1, 2, 3 \rangle}$

3. Surface defined by $F(x,y,z) = 0$ where

$$F(x,y,z) = yz - \ln(x+z)$$

- $F_x = -\frac{1}{x+z}$, $F_y = z$, $F_z = y - \frac{1}{x+z}$

Tangent Plane:

- $\nabla F(0,0,1) = \langle -\frac{1}{0+1}, 1, 0 - \frac{1}{0+1} \rangle = \langle -1, 1, -1 \rangle$

$$\nabla F(0,0,1) \cdot \langle x-0, y-0, z-1 \rangle = 0$$

$$\langle -1, 1, -1 \rangle \cdot \langle x, y, z-1 \rangle = 0$$

$$\boxed{-x + y - (z-1) = 0}$$

(tangent plane)
Note, it is an equation.

Normal line:

$$L(t) = \langle 0,0,1 \rangle + t(\nabla F(0,0,1))$$

$$= \langle 0,0,1 \rangle + t\langle -1, 1, -1 \rangle$$

$$= \boxed{\langle -t, t, 1-t \rangle}$$

Normal line

$$\boxed{x(t) = -t, \quad y(t) = t, \quad z(t) = 1-t}$$

4. $f_x = 6x^2 + y^2 + 10x$

$f_y = 2xy + 2y$

1. Find crit pts:

$\underline{f_y = 0}$: $2y(x+1) = 0$

$y = 0$ or $x = -1$.

$\underline{f_x = 0}$: $6x^2 + 10x + y^2 = 0$

If $y = 0$:

$6x^2 + 10x = 0$

$2x(3x+5) = 0$

$x = 0$ or $x = -5/3$

If $x = -1$:

$6 - 10 + y^2 = 0$

$y^2 = 4$

$y = \pm 2$

So, the critical points are

$$(0,0),\ (-5/3, 0),\ (-1,-2),\ (-1,2).$$

② Classify: $f_{xx} = 12x+10$, $f_{xy} = 2y$, $f_{yy} = 2x+2$

$$D = f_{xx} f_{yy} - [f_{xy}]^2 = (12x+10)(2x+2) - [2y]^2$$

$$= 4(6x+5)(x+1) - 4y^2$$

- $D(0,0) = 4 \cdot 5 \cdot 1 - 0 = 20 > 0$. $f_{xx}(0,0) = 10 > 0$. So

  and $(0,0)$ is a local min.

- $D(-5/3, 0) = 4(-5)(-\frac{2}{3}) > 0$. $f_{xx}(-5/3, 0) = -20+10 = -10 < 0$.

  So and $(-5/3, 0)$ is a local max.

- $D(-1,-2) = 4(-1)(0) - 4 \cdot 4 = -16 < 0$, so $(-1,-2)$ is a saddle pt.
- $D(-1,2) = 4(-1)(0) - 4 \cdot 4 = -16 < 0$ so $(-1,2)$ is a saddle pt.

5. Distance $\sqrt{x^2+y^2+z^2}$ is minimized when $x^2+y^2+z^2$ is minimized.

- Let $w = x^2+y^2+z^2 = x^2+(9+xz)+z^2$ (for points on surface)

- Find critical points of w:

$$w_x = 2x + z$$
$$w_z = x + 2z.$$

- $w_x = 0$ and $w_z = 0$ give a linear system:

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So, the only solution is $(x,z) = (0,0)$; this is the only critical point of w.

- Although we know from the problem set up that the points with minimum distance give critical points for w, we can verify $(0,0)$ is a local min: ~~0~~ $w_{xx} = 2$, $w_{xz} = 1$, $w_{zz} = 2$

~~To~~ $D = w_{xx}w_{zz} - [w_{xz}]^2 = 2\cdot2 - 1^2 = 3 > 0$

So $(0,0)$ is a local extremum. Since $w_{xx} = 2 > 0$, it is a local min 

, as expected.

- From the problem, we know an absolute minimum exists; it must also be a local min, so the abs. min must be at $(x, z) = (0, 0)$. The corresponding y values are

$$y^2 = 9 + xz$$

$$y^2 - 9 = 0$$

$$(y+3)(y-3) = 0$$

$$y = \pm 3$$

- So, the points on $y^2 = 9 + xz$ closest to the origin are $\boxed{(0, -3, 0) \text{ and } (0, 3, 0)}$