Name: Solutions

Directions: Show all work. No credit for answers without work. This test has 110 points but scores will be taken out of 100.

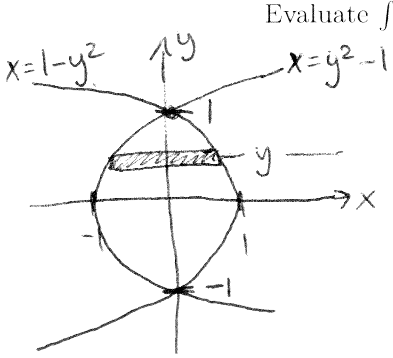
1. **[15 points]** Evaluate $\int_0^{\pi/2}\int_0^1 (\sin x)e^y\,dy\,dx$.

$$= \left(\int_0^{\pi/2} \sin x\,dx\right)\left(\int_0^1 e^y\,dy\right) = (-\cos x)\Big|_0^{\pi/2} \cdot e^y\Big|_0^1$$

$$= \left((-0)-(-1)\right)\cdot\left(e^1 - e^0\right)$$

$$= 1 \cdot \left(e^1 - e^0\right) = \boxed{e-1}$$

2. **[15 points]** Let D be the region bounded by the parabolas $x = 1 - y^2$ and $x = y^2 - 1$. Evaluate $\iint_D y^2\,dA$.



- $1-y^2 = y^2-1 \iff 2y^2=2 \iff y^2=1,\ y=\pm 1$

- Use horizontal Rectagles.

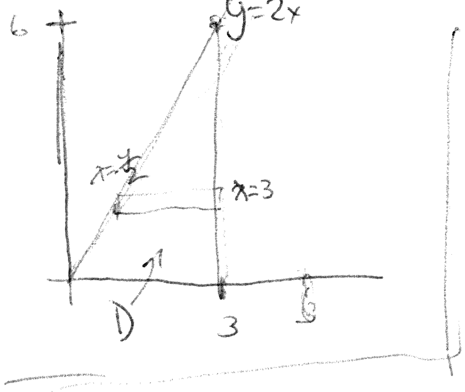
- $\iint_D y^2\,dA = \int_{-1}^{1}\int_{y^2-1}^{1-y^2} y^2\,dx\,dy = \int_{-1}^{1} y^2 x\Big|_{x=y^2-1}^{x=1-y^2} dy$

$$= \int_{-1}^{1} y^2\left((1-y^2)-(y^2-1)\right)dy = \int_{-1}^{1} y^2(2-2y^2)\,dy = 2\int_{-1}^{1} y^2(1-y^2)\,dy$$

$$= 2\int_{-1}^{1} y^2 - y^4\,dy = 2\left(\frac{y^3}{3}-\frac{y^5}{5}\right)\Big|_{-1}^{1} = 2\left(\left(\frac{1}{3}-\frac{1}{5}\right)-\left(\frac{-1}{3}-\frac{-1}{5}\right)\right)$$

$$= 4\left(\frac{1}{3}-\frac{1}{5}\right) = 4\left(\frac{5}{15}-\frac{3}{15}\right) = \boxed{\frac{8}{15}}$$

3. **[15 points]** Evaluate $\int_0^6 \int_{y/2}^3 y\cos(1+x^3)\,dx\,dy$. Hint: interpret as a double integral over a region and change the order of integration.



$$= \iint_D y\cos(1+x^3)\,dA$$

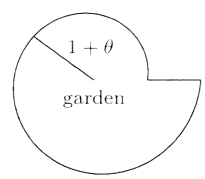
$$= \int_0^3 \int_0^{2x} y\cos(1+x^3)\,dy\,dx$$

$$= \int_0^3 \frac{y^2}{2}\cos(1+x^3)\Big|_{y=0}^{y=2x}\,dx$$

$$= \int_0^3 \frac{4x^2}{2}\cos(1+x^3)\,dx \quad = \int_1^{28} \cos u \cdot \frac{2}{3}\,du \quad = \frac{2}{3}\sin u\Big|_{u=1}^{u=28}$$

$$u = 1+x^3,\ du = 3x^2\,dx \qquad = \boxed{\frac{2}{3}\sin(28) - \frac{2}{3}\sin(1)}$$

4. **[15 points]** A spiral-shaped fence encloses a modern garden containing the origin $(0,0)$. The fence is described by the polar equation $r = 1+\theta$ for $0 \le \theta \le 2\pi$. Find the area of the garden.



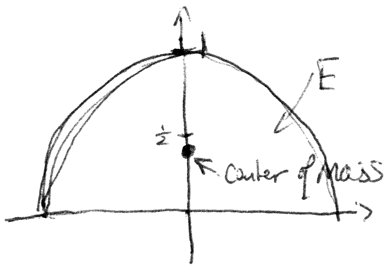
$$\text{Area} = \iint_E dA \quad = \int_0^{2\pi}\int_0^{1+\theta} r\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{r^2}{2}\Big|_{r=0}^{r=1+\theta}\,d\theta$$

$$= \int_0^{2\pi} \frac{(1+\theta)^2}{2}\,d\theta$$

$$= \frac{(1+\theta)^3}{6}\Big|_{\theta=0}^{\theta=2\pi} = \boxed{\frac{(2\pi+1)^3}{6} - \frac{1}{6}}$$

5. [**15 points**] A lamina of uniform density occupies the semi-circular region D consisting of all points (x, y) such that $x^2 + y^2 \leq 1$ and $y \geq 0$. Find the center of mass.



- $\rho(x,y) = K$ for some const. K.
- $m = \iint_E K\,dA = K\int_0^\pi \int_0^1 r\,dr\,d\theta$

$$= K\int_0^\pi \frac{r^2}{2}\Big|_{r=0}^{r=1} d\theta = K\int_0^\pi \frac{1}{2}d\theta$$

$$= K\frac{\pi}{2}$$

- $M_x = \iint_E y\,\rho(x,y)\,dA = \int_0^\pi \int_0^1 y \cdot K \cdot r\,dr\,d\theta$

$$= K\int_0^\pi \int_0^1 r^2 \sin\theta\,dr\,d\theta = K\left(\int_0^\pi \sin\theta\,d\theta\right)\left(\int_0^1 r^2 dr\right)$$

$$= K\left(-\cos\theta\Big|_{\theta=0}^{\theta=\pi}\right)\cdot\left(\frac{r^3}{3}\Big|_{r=0}^{r=1}\right) = K\big((1)-(-1)\big)\cdot\frac{1}{3} = \frac{2}{3}K$$

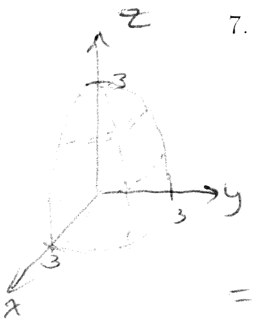
- $M_y = \iint_E x\,\rho(x,y)\,dA = 0$, by Symmetry.

- $CoM = \left(\frac{M_y}{m}, \frac{M_x}{m}\right) = \left(0, \frac{2}{3}K\cdot\frac{2}{K\pi}\right) = \boxed{\left(0, \frac{4}{3\pi}\right)} \approx (0, 0.424)$

6. [**5 points**] What does it mean for a vector field to be conservative?

A vector field F is conservative if it is the gradient of another function f, i.e. $F = \nabla f$.

7. **[15 points]** Evaluate $\iiint_E e^{(x^2+y^2+z^2)^{3/2}}\,dV$, where E is ~~enclosed~~ The solid bounded by the sphere $x^2+y^2+z^2=9$ in the first octant (where x, y, and z are all at least zero).



$$\int_0^{\pi/2}\int_0^{\pi/2}\int_0^3 e^{(x^2+y^2+z^2)^{3/2}}\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi \qquad d\rho\,d\theta\,d\phi$$

$$=\int_0^{\pi/2}\int_0^{\pi/2}\int_0^3 e^{(\rho^2)^{3/2}}\rho^2\sin\phi\,d\rho\,d\theta\,d\phi = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^3 \rho^2 e^{\rho^3}\sin\phi$$

$$=\left(\int_0^3 \rho^2 e^{\rho^3}\,d\rho\right)\cdot\left(\int_0^{\pi/2}d\theta\right)\cdot\left(\int_0^{\pi/2}\sin\phi\,d\phi\right)$$

$u=\rho^3$
$du=3\rho^2 d\rho$

$$=\left(\int_0^{27} e^u\cdot\tfrac{1}{3}du\right)\cdot \pi/2\cdot(-\cos\phi)\Big|_0^{\pi/2} = \tfrac{1}{3}e^u\Big|_0^{27}\cdot\frac{\pi}{2}\cdot((-0)-(-1))$$

$$=\tfrac{1}{3}(e^{27}-1)\cdot\frac{\pi}{2}\cdot 1 = \boxed{\frac{\pi}{6}(e^{27}-1)}$$

8. **[15 points]** Evaluate $\int_C xy^2\,ds$ where C is the curve given by $\vec{r}(t)=(2\sin t)\vec{i}+(2\cos t)\vec{j}+3t\vec{k}$, for $0\le t\le\pi$.

- $\vec{r}\,'(t)=\langle 2\cos t, -2\sin t, 3\rangle$; $|\vec{r}\,'(t)|=\sqrt{4\cos^2 t+4\sin^2 t+9}=\sqrt{13}$

- $\int_C xy^2\,ds = \int_0^{\pi}(2\sin t)(2\cos t)^2\,|\vec{r}\,'(t)|\,dt = 8\sqrt{13}\int_0^{\pi}(\cos^2 t)(\sin t)\,dt$

$u=\cos t$
$du=-\sin t\,dt$

$$=8\sqrt{13}\cdot\int_1^{-1} u^2(-du)$$

$$=8\sqrt{13}\int_{-1}^{1}u^2\,du$$

$$=8\sqrt{13}\;\frac{u^3}{3}\Big|_{-1}^{1}$$

$$=8\sqrt{13}\left(\frac{1}{3}-\frac{(-1)^3}{3}\right)=8\sqrt{13}\cdot\frac{2}{3}$$

$$=\boxed{\frac{16\sqrt{13}}{3}}$$