Name: Solutions

Directions: Show all work. No credit for answers without work.

1. **[2.5 points]** Calculate $\int_1^2 \int_0^1 (x+y)^{-2}\, dx\, dy$.

$$= \int_1^2 \left[-(x+y)^{-1} \right]_{x=0}^{x=1} dy$$

$$= \int_1^2 \left[\left(-\frac{1}{1+y}\right) - \left(-y^{-1}\right) \right] dy$$

$$= \int_1^2 \frac{1}{y} - \frac{1}{1+y}\, dy$$

$$= \left[\ln(y) - \ln(1+y) \right]_{y=1}^{y=2} = \left[\ln(2) - \ln(3)\right] - \left[\ln(1) - \ln(2)\right]$$

$$= 2\ln(2) - \ln(3) = \boxed{\ln\left(\frac{4}{3}\right)}$$

2. **[2.5 points]** Find the volume of the solid that lies under the plane $3x + 5y + z = 12$ and above the rectangle $R = \{(x, y) \colon\ 0 \le x \le 1,\ 0 \le y \le 1\}$.

$$V = \iint_R 12 - 3x - 5y\, dA = \int_0^1 \int_0^1 12 - 3x - 5y\, dy\, dx$$

$$= \int_0^1 \left[(12-3x)y - \frac{5}{2}y^2 \right]_{y=0}^{y=1} dx$$

$$= \int_0^1 \left[(12-3x)1 - \frac{5}{2} \right] - [0]\, dx$$

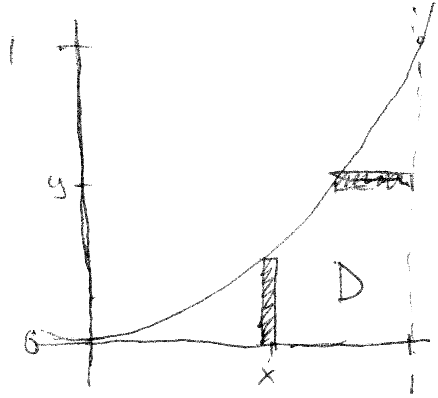
$$= \int_0^1 \frac{19}{2} - 3x\, dx = \left[\frac{19}{2}x - \frac{3}{2}x^2 \right]_{x=0}^{x=1}$$

$$= \left(\frac{19}{2} - \frac{3}{2} \right) - 0 = \frac{16}{2} = \boxed{8}$$

3. [**2.5 points**] Evaluate $\int_0^1 \int_{\sqrt{y}}^1 \sqrt{x^3+1}\, dx\, dy$.

$y = x^2, \; x = \sqrt{y}$

- For each y, we integrate from $x = \sqrt{y}$ to $x = 1$.
- So, for each y, we ~~integrate~~ $y = x^2$ to $x = 1$



$$= \int_0^1 \int_0^{x^2} \sqrt{x^3+1}\, dy\, dx$$

$$= \int_0^1 \left[\left(\sqrt{x^3+1}\right) y\right]_{y=0}^{y=x^2} dx$$

$$= \int_0^1 x^2 \sqrt{x^3+1}\, dx \quad , \quad u = x^3+1; \quad du = 3x^2\, dx;$$

$$= \int_1^2 u^{1/2} \cdot \frac{1}{3}\, du = \frac{1}{3} \cdot \frac{2}{3} u^{3/2} \Big|_{u=1}^{u=2} = \frac{2}{9}\left(2^{3/2} - 1^{3/2}\right) = \boxed{\frac{4\sqrt{2} - 2}{9}}$$

4. [**2.5 points**] Let D be the unit disc; that is, $D = \{(x, y) \colon 0 \le x^2 + y^2 \le 1\}$. Evaluate $\iint_D x^2 \sqrt{x^2 + y^2}\, dA$.

$$= \int_0^{2\pi} \int_0^1 (r\cos\theta)^2 \cdot r \cdot r\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^1 r^4 \cos^2\theta\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[\frac{\cos^2\theta}{5} \cdot r^5\right]_{r=0}^{r=1} d\theta = \int_0^{2\pi} \frac{1}{5}\cos^2\theta\, d\theta$$

$$= \int_0^{2\pi} \frac{1 + \cos(2\theta)}{10}\, d\theta = \left[\frac{1}{10}\theta + \frac{1}{20}\sin(2\theta)\right]_{\theta=0}^{\theta=2\pi}$$

$$= \left[\frac{\pi}{5} + 0\right] - \left[0 + 0\right] = \boxed{\frac{\pi}{5}}$$