Name: Solutions

**This test has 60 points (~~10 points per page~~) but is scored out of 50 points. Scores are truncated at 50.**

1. **[4 points]** A survey is conducted on television advertisements. A total of 15 commercials used music, 11 displayed text, and 12 used a narrator. Also, 2 commercials used a narrator and music, 5 used a narrator and text, and 3 used music and text. Finally, 2 commercials used music, text, and a narrator, and 1 commercial used none of these. In total, how many commercials are in the survey?

$A_1$: music

$A_2$: text

$A_3$: narrator

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$$

$$= 15 + 11 + 12 - 3 - 2 - 5 + 2$$

$$= 30$$

All commercials: $30 + 1 = \boxed{31}$

2. **[2 parts, 3 points each]** A trade school offers 50 classes. Every student takes 4 classes each semester.

   (a) How many students must the school have to guarantee that there are two students with exactly the same course schedule?

   $\#$ course schedules $= C(50,4)$

   $\#$ to guarantee two have same schedule: $\boxed{C(50,4) + 1}$ $= \frac{50 \cdot 49 \cdot 48 \cdot 47}{4 \cdot 3 \cdot 2 \cdot 1} + 1 = \boxed{230{,}301}$

   (b) Suppose that the school has 1827 full time students. What can you say about the number of students registered for the largest class?

   - Total $\#$ (student, class) pairs $\geq 4 \cdot 1827 = 7{,}308$
   - Pigeonhole: largest class has $\geq \frac{7308}{50}$ 146.16 students

   $\boxed{\text{so largest class has} \geq 147 \text{ students}}$

3. [**2 points**] State the mathematical relationship between $C(n,k)$ and $P(n,k)$.

$$\boxed{P(n,k) = C(n,k)\cdot k!}$$

4. [**2 points**] State the binomial theorem.

$$\boxed{(x+y)^n = \sum_{i=0}^{n} C(n,i)\cdot x^{n-i}\cdot y^i}$$

5. Find the following coefficients. Except in part (a), you may leave your answer in terms of permutation numbers (e.g. $P(n,r)$), binomial coefficients (e.g. $C(n,r)$), and factorials (e.g. $n!$).

(a) [**2 points**] Find the numerical value of the coefficient of $x^8$ in $(x-2)^{12}$.

Want: $i=4$. $\quad C(12,4)\cdot x^8\cdot(-2)^4 = 16\cdot C(12,4)\cdot x^8$

$$= 16\cdot 495\cdot x^8$$
$$= \boxed{7920}\,x^8$$

(b) [**2 points**] Find the coefficient of $x^3y^6$ in $(2x+3y)^9$.

$\cancel{\boxed{C(9,3)}\cdot x^3 y^6}$

$$C(9,6)\cdot(2x)^3\cdot(3y)^6$$
$$= \boxed{C(9,6)\cdot 2^3\cdot 3^6}\, x^3y^6$$

or $\boxed{C(9,3)\cdot 2^3\cdot 3^6}\, x^3y^6$

(c) [**2 points**] Find the coefficient of $x^4y$ in $(3x-y+2)^{14}$.

$A=3x$

$B=-y$

$C=2$

$$\frac{14!}{4!\,1!\,9!}(3x)^4\cdot(-y)^1\cdot(2)^9 = \boxed{-3^4\cdot 2^9\cdot\frac{14!}{4!\,1!\,9!}}\,x^4y$$

6. [**6 points**] Use the Euclidean algorithm to find gcd(1734, 1628) and express it as a linear combination of 1734 and 1628. Show your work.

$1734 = 1\cdot 1628 + 106$

$1628 = 15\cdot 106 + 38$

$106 = 2\cdot 38 + 30$

$38 = 1\cdot 30 + 8$

$30 = 3\cdot 8 + 6$

$8 = 1\cdot 6 + 2$

$6 = 3\cdot 2 + 0$

$\underline{\gcd(2,0) = 2}.$

$2 = 8 - 1\cdot 6$

$2 = 8 - 1\cdot(30 - 3\cdot 8)$

$2 = 4\cdot 8 - 1\cdot 30$

$2 = 4\cdot(38 - 1\cdot 30) - 1\cdot 30$

$2 = 4\cdot 38 - 5\cdot 30$

$2 = 4\cdot 38 - 5\cdot(106 - 2\cdot 38)$

$2 = 14\cdot 38 - 5\cdot 106$

$2 = 14\cdot(1628 - 15\cdot 106) - 5\cdot 106$

$2 = 14\cdot 1628 - 215\cdot 106$

$2 = 14\cdot 1628 - 215\cdot(1734 - 1\cdot 1628)$

$\boxed{2 = 229\cdot 1628 - 215\cdot 1734}$

7. [**4 points**] How many numbers in $\{1, 2, \ldots, 999\}$ are relatively prime to 1000?

$1000 = (10)^3 = 2^3\cdot 5^3$

$\varphi(1000) = 1000\cdot\left(1-\frac{1}{2}\right)\cdot\left(1-\frac{1}{5}\right) = 1000\cdot\frac{1}{2}\cdot\frac{4}{5} = \boxed{400}$

8. [**4 points**] Give an example of a relation on $\{1,2,3\}$ that is reflexive, symmetric, and <u>not</u> transitive.

$$\rho = \{(1,1),\ (2,2),\ (3,3),\ (1,2),\ (2,3),\ (2,1),\ (3,2)\}$$

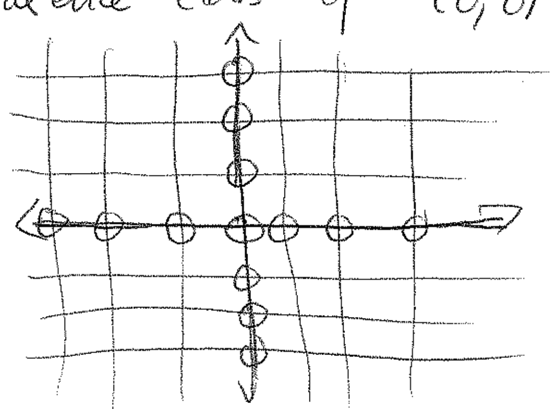
or $\quad x \,\rho\, y \iff |x-y| \geq 1$

(other answers are possible.)

9. [**4 points**] Consider the equivalence relation $\rho$ on $\mathbb{Z} \times \mathbb{Z}$ defined by $(x_1, y_1)\ \rho\ (x_2, y_2) \leftrightarrow x_1 y_1 = x_2 y_2$. Which ordered pairs in $\mathbb{Z} \times \mathbb{Z}$ are in the equivalence class of $(0,0)$? Describe the equivalence class of $(0,0)$.

$$(x,y)\,\rho\,(0,0) \iff xy = 0 \cdot 0$$
$$\iff xy = 0.$$

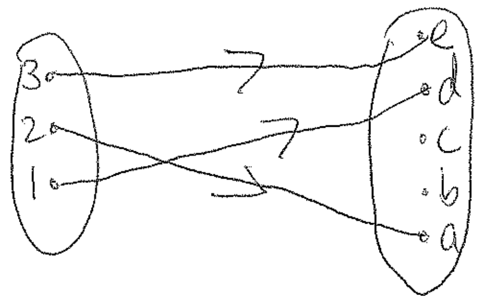
So equivalence class of $(0,0) = \{(x,y) : x = 0 \text{ or } y = 0\}$



Circled points.

10. [**4 parts, 3 points each**]

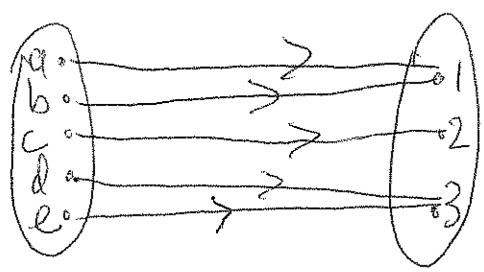
(a) Give an example of a function from $\{1,2,3\}$ to $\{a,b,c,d,e\}$ which is one-to-one/injective but not surjective/onto.




(b) How many one-to-one/injective functions are there from $\{1,2,3\}$ to $\{a,b,c,d,e\}$?

$$\boxed{5\cdot 4\cdot 3} = \boxed{60}$$

(c) Give an example of an onto/surjective function from $\{a,b,c,d,e\}$ to $\{1,2,3\}$.




(d) How many onto/surjective functions from $\{a,b,c,d,e\}$ to $\{1,2,3\}$ are there? Hint: count the complement. Let $A_1$ be the set of functions that map nothing to 1. Let $A_2$ be the set of functions that map nothing to 2. Let $A_3$ be the set of functions that map nothing to 3. What is $|A_1 \cup A_2 \cup A_3|$?

Total # of functions: $3^5$

# that are not onto: $|A_1 \cup A_2 \cup A_3| = |A_1|+|A_2|+|A_3|$

$- |A_1 \cap A_2| - |A_1 \cap A_3| - |A_1 \cap A_2|$

$+ |A_1 \cap A_2 \cap A_3|$

$= 3\cdot 2^5 - 3\cdot 1^5 + 0 = 3\cdot 32 - 3$

$= 3\cdot 31 = 93$

Total # onto: $3^5 - 93 = \boxed{150}$ .5

11. **[2 points]** A 6-slot database uses a hashing strategy to store numbers; the hash function is $h(x) = x \mod 6$. Initially, the database is empty. Show a picture of the hash table after the numbers 843, 145, 1932, 533, 204 are inserted in the given order. Collisions are resolved by chaining.

| 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1932 → 204 | 145 | | 843 | | 533 |

12. **[2 parts, 4 points each]** In the RSA algorithm, let $p = 47$ and $q = 113$, so that $n = 5311$ and $\varphi(n) = 5152$. Pick $e = 13$.

(a) Use the Euclidean algorithm to find $d$. Show your work.

$$5152 = 396 \cdot 13 + 4$$
$$13 = 3 \cdot 4 + 1$$
$$4 = 4 \cdot 1 + 0$$
$$\gcd = 1$$

$$1 = 13 - 3 \cdot 4$$
$$1 = 13 - 3(5152 - 396 \cdot 13)$$
$$1 = 1189 \cdot 13 - 3 \cdot 5152$$
$$d = 1189 \mod 5752 = \boxed{1189}$$

(b) Encode the plaintext message $T = 1024$. Show your work.

$$U = T^e \mod n$$
$$U = (1024)^{13} \mod 5311$$

All mod 5311:

$$(1024)^2 = 2309$$
$$(1024)^4 = (2309)^2 = 4548$$
$$(1024)^8 = (4548)^2 = 3270$$

$$(1024)^{12} = (1024)^8 \cdot (1024)^4$$
$$= 3270 \cdot 4548$$
$$= 1160$$
$$(1024)^{13} = 1160 \cdot 1024$$
$$= \boxed{3487}$$