Name: ______________________________

This test has 60 points (10 points per page) but is scored out of 50 points. Scores are truncated at 50.

1. [**4 points**] Express $1 + 4 + 9 + \cdots + n^2$ in sigma summation notation ($\sum$).

2. [**6 points**] Using induction, prove that $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$ for $n \geq 1$.

3. Consider the following code fragment.

```
DoSomething(n):
  j = 0
  i = 1
  while i ≤ n do
      j = j + 2i − 1
      i = i + 1
  end while
  write j
```

(a) [**2 points**] What number does DoSomething(2) write?

(b) [**2 points**] What number does DoSomething(3) write?

(c) [**2 points**] What number does DoSomething(n) write?

(d) [**4 points**] Find a loop invariant that would allow you to prove your previous answer is correct. (You do not need to prove that the condition you provide is a loop invariant.)

4. [**3 points**] A collection of strings S over the alphabet $\{a, b\}$ is defined recursively as follows. Write down the 4 shortest strings in S.

   - S contains the empty string λ.
   - If $x \in S$, then $axxb \in S$.

5. Let $A = \{1, \{5\}, 5, 6, 7\}$, $B = \{\varnothing, \{3\}, 4, 5, 6\}$, and $C = \{\varnothing, \{4, 6\}\}$.

   (a) [**6 parts, 0.5 points each**] True or False? (Write the whole word as your answer.)

   i. $\{6\} \in A$

   ii. $\{\varnothing, \{3\}\} \subseteq B$

   iii. $\{5, \{5\}\} \in A$

   iv. $\{5, \{5\}\} \subseteq A$

   v. $\{4, 6\} \in C$

   vi. $\{4, 6\} \subseteq C$

   (b) [**2 points**] Find $B \cap C$.

   (c) [**2 points**] Find the powerset $\mathcal{P}(C)$.

6. Let $T(n) = T(n-1) + 30T(n-2)$ for $n \geq 3$, $T(1) = 1$, and $T(2) = 1$.

   (a) [**1 point**] Find the first four values of $T(n)$, from $T(1)$ through $T(4)$.

   (b) [**4 points**] Solve the recurrence.

7. Let $T(n) = 3T(n-1) + 7$ for $n \geq 2$, $T(1) = -1$.

   (a) [**1 point**] Find the first four values of $T(n)$, from $T(1)$ through $T(4)$.

   (b) [**4 points**] Solve the recurrence.

8. [**5 points**] Let A and B be infinite, countable sets. Is $A \cup B$ always countable? Show that your answer is correct.

9. [**5 points**] Let S be the collection of all infinite strings over the alphabet $\{a, b\}$. For example, the string $aaaa\cdots$ consisting of all a's, the string $abab\cdots$ of alternating a's and b's are both members of S. Is S countable? Show that your answer is correct.

10. [**3 points**] How many 4-digit ATM pins have first and last digits that are both even? For example, 0760 and 8352 count, but 1234 and 3221 do not. Show your work.

11. [**3 points**] How many 4-digit ATM pins contain exactly one 0? For example, 3021 and 0988 count, but 2010 and 3113 do not. Show your work.

12. [**4 points**] How many 4-digit ATM pins use each digit at most twice? For example, 2127, 5566, and 1234 count, but 4544 and 9999 does not. Show your work.