Name: Solutions

This test has 60 points (10 points per page) but is scored out of 50 points. Scores are truncated at 50.

1. [**4 points**] Express $1 + 4 + 9 + \cdots + n^2$ in sigma summation notation ($\sum$).

$$\sum_{j=1}^{n} j^2 = n^2$$

2. [**6 points**] Using induction, prove that $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$ for $n \geq 1$.

Proof: By induction on n.

Basis Step: If $n=1$, then $1 = \frac{1(1+1)}{2}$, so the equation holds.

Inductive Step: Suppose $n \geq 2$. By IH,

$$1 + \cdots + n-1 = \frac{(n-1)(n)}{2}.$$

Adding n to both sides gives

$$1 + \cdots + n-1 + n = \frac{(n-1)n}{2} + n.$$

The RHS simplifies: $\frac{(n-1)n}{2} + \frac{2n}{2} = \frac{(n-1)n+2n}{2}$

$$= \frac{n(n+1)}{2}.$$

Therefore $1 + \cdots + n = \frac{n(n+1)}{2}$. $\blacksquare$

3. Consider the following code fragment.

```
DoSomething(n):
  j = 0
  i = 1
  while i ≤ n do
      j = j + 2i − 1
      i = i + 1
  end while
  write j
```

(a) [**2 points**] What number does DoSomething(2) write?

$j = \cancel{0}\ \cancel{1}\ 4$

$i = \cancel{1}\ \cancel{2}\ 3$

$\boxed{4}$

(b) [**2 points**] What number does DoSomething(3) write?

$4 + 2\cdot 3 - 1 = \boxed{9}$

(c) [**2 points**] What number does DoSomething(n) write?

$\boxed{n^2}$

(d) [**4 points**] Find a loop invariant that would allow you to prove your previous answer is correct. (You do not need to prove that the condition you provide is a loop invariant.)

$\boxed{Q:\ j = (i-1)^2}$

4. **[3 points]** A collection of strings S over the alphabet $\{a, b\}$ is defined recursively as follows. Write down the 4 shortest strings in S.

- S contains the empty string λ.
- If $x \in S$, then $axxb \in S$.

λ, ab, a(ab)(ab)b, a(aababb)(aababb)b

$\boxed{\lambda, \text{ab}, \text{aababb}, \text{aaababbaababbb}}$

5. Let $A = \{1, \{5\}, 5, 6, 7\}$, $B = \{\varnothing, \{3\}, 4, 5, 6\}$, and $C = \{\varnothing, \{4, 6\}\}$.

(a) **[6 parts, 0.5 points each]** True or False? (Write the whole word as your answer.)

i. $\{6\} \in A$

FALSE

ii. $\{\varnothing, \{3\}\} \subseteq B$

True

iii. $\{5, \{5\}\} \in A$

FALSE

iv. $\{5, \{5\}\} \subseteq A$

True

v. $\{4, 6\} \in C$

True

vi. $\{4, 6\} \subseteq C$

FALSE

(b) **[2 points]** Find $B \cap C$.

$B \cap C = \{\varnothing\}$

(c) **[2 points]** Find the powerset $\mathcal{P}(C)$.

$\mathcal{P}(C) = \boxed{\{\varnothing, \{\varnothing\}, \{\{4,6\}\}, \{\varnothing, \{4,6\}\}\}}$

6. Let $T(n) = T(n-1) + 30T(n-2)$ for $n \geq 3$, $T(1) = 1$, and $T(2) = 1$.

(a) [**1 point**] Find the first four values of $T(n)$, from $T(1)$ through $T(4)$.

$1, 1, 31, 61$

(b) [**4 points**] Solve the recurrence.

$t^2 = t + 30$

$t^2 - t - 30 = 0$

$(t-6)(t+5) = 0$

$t = 6$ or $t = -5$.

$\underline{T(1)}$: $1 = p + q$

$\underline{T(2)}$: $1 = 6p - 5q$

$5 = 5p + 5q$

$6 = 11p$

$p = \frac{6}{11}$, $q = 1 - p = \frac{5}{11}$

$T(n) = p6^{n-1} + q(-5)^{n-1}$

$$\boxed{T(n) = \frac{6}{11} \cdot 6^{n-1} + \frac{5}{11} \cdot (-5)^{n-1}}$$

7. Let $T(n) = 3T(n-1) + 7$ for $n \geq 2$, $T(1) = -1$.

(a) [**1 point**] Find the first four values of $T(n)$, from $T(1)$ through $T(4)$.

$-1, 4, 19, 64$

(b) [**4 points**] Solve the recurrence.

$$T(n) = c^{n-1} T(1) + \sum_{i=2}^{n} c^{n-i} g(i)$$

$$= 3^{n-1} \cdot (-1) + \sum_{i=2}^{n} 3^{n-i} \cdot 7$$

$$= -3^{n-1} + 7(3^{n-2} + 3^{n-3} + \dots + 1)$$

$$= -3^{n-1} + 7 \cdot \frac{3^{n-1} - 1}{3 - 1} = -3^{n-1} + \frac{7}{2}(3^{n-1} - 1)$$

$$= \boxed{\frac{5}{2} \cdot 3^{n-1} - \frac{7}{2}}$$

$$= \boxed{\frac{5}{6} \cdot 3^{n} - \frac{7}{2}}$$

8. **[5 points]** Let A and B be infinite, countable sets. Is $A \cup B$ always countable? Show that your answer is correct.

Yes, $A \cup B$ is countable. Since A is countable, we can list its elements in some order, so that each elt. appears: $a_1, a_2, a_3, a_4, \ldots$. Similarly, we can list the elements of B: $b_1, b_2, b_3, b_4, \ldots$.

To list the elements of $A \cup B$, we alternate between these lists (crossing out any duplicate entries):

$$a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4, \ldots$$

Therefore $A \cup B$ is countable. ∎

9. **[5 points]** Let S be the collection of all infinite strings over the alphabet $\{a, b\}$. For example, the string $aaaa\cdots$ consisting of all a's, the string $abab\cdots$ of alternating a's and b's are both members of S. Is S countable? Show that your answer is correct.

No. Suppose for a contradiction that S is countable. Then we could list the elements of S as $s_1, s_2, \ldots$ and organize these strings in a table, for example:

s_1:	a	a	b	b	a	a	b
s_2:	b	a	b	a	b	a	b
s_3:	b	b	b	b	b	b	b
s_4:	a	a	a	b	a	a	a

Bad string:

$x = bbaa$

Using Cantor's diagonalization technique, we construct a string x not in the list by choosing the j^{th} character of x to differ from the j^{th} character of s_j. Hence, S is not countable. ∎

10. [**3 points**] How many 4-digit ATM pins have first and last digits that are both even? For example, 0760 and 8352 count, but 1234 and 3221 do not. Show your work.

Use product principle:

1 st digit: 5 choices
2nd ": 10
3rd ": 10
4th ": × 5

$\boxed{2500}$

11. [**3 points**] How many 4-digit ATM pins contain exactly one 0? For example, 3021 and 0988 count, but 2010 and 3113 do not. Show your work.

Use addition principle:

#pins with exactly one 0 in first spot:

1st digit: 1 choice
2nd ": 9 choices
3rd ": 9
4th ": 9

729

Since the 0 can also occur in the 2nd, 3rd, and 4th position:

total #: 729+729+729+72

$= \boxed{2916}$

12. [**4 points**] How many 4-digit ATM pins use each digit at most twice? For example, 2127, 5566, and 1234 count, but 4544 and 9999 does not. Show your work.

Count complement: How many pins use some digit ≥ 3 times?

⇒ Use product principle. and addition principle:

- 10 pins where some digit occurs 4 times
- 360 pins where some digit occurs exactly 3 times:

– Choose digit that happens 3 times
– Choose digit that occurs once: (9
– Choose where the single digit goes: ×4

360

- Total # bad pins = 370
- #good = 10,000 − 370 = $\boxed{9630}$