Name: ____________________

Do not turn the page until instructed.

Directions:

1. Write your name on this page.

2. Show your work unless instructed otherwise. Answers without supporting work are incomplete and are marked accordingly.

3. You may use a calculator.

4. You may use a single 8.5×11-inch paper with handwritten notes.

5. Turn off and put away all other electronic equipment (especially cell phones), notes, and books.

6. The exam is scored out of 115 points, but the questions are worth 130 points in total.

7. Good luck!

1. [**6 points**] Truth tables.

   (a) Write a truth table for the following wff: $(A \to B) \wedge (B \to C) \wedge (C \to A) \to (A \vee C')$.

   (b) Is the above wff a tautology? Why or why not?

2. [**4 parts, 1 point each**] Write the negation of the following sentences as naturally as possible.

   (a) The weather is dry and cold.

   (b) If John likes Sarah, then Sarah is happy.

   (c) Every dog chases cats.

   (d) John is happy if and only if everyone likes him.

3. [**4 parts, 1 point each**] Determine the truth value of the following wffs in the interpretation where the domain is $\{1, 2, 3, \ldots\}$, $Q(x)$ is "x is a multiple of 3", and $P(x)$ is "x is prime". For each wff, you should answer "true" or "false". You do not need to show your work.

   (a) $\forall x\ [Q(x) \to (P(x))'\,]$

   (b) $\exists x\ [P(x) \to Q(x)]$

   (c) $\forall x\ [\exists y\ [Q(y) \wedge\ (y \leq x)\,]]$

   (d) $\exists x\ [\forall y\ [\,(P(y) \wedge Q(y))\ \to\ (x = y)\,]]$

4. [**4 parts, 1 point each**] Translate the following English sentences to wffs using the given predicates; the domain is the set of all people.

$A(x)$: "x is an astronaut"	$P(x)$: "x is a politician"
$L(x, y)$: "x likes y"	$K(x, y)$: "x knows y"

   (a) Some people like only people they know.

   (b) Some astronauts are also politicians.

   (c) Nobody likes every politician.

   (d) Some astronauts are liked by everyone.

5. [**5 points**] Let x be an integer. Prove that x is even if and only if x^2 is even. (You may assume that every integer is even or odd, but not both.)

6. [**5 points**] Prove by induction that $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ for every positive integer n.

7. [**5 points**] The following algorithm returns the maximum value in an array of positive integers. Find the loop invariant Q.

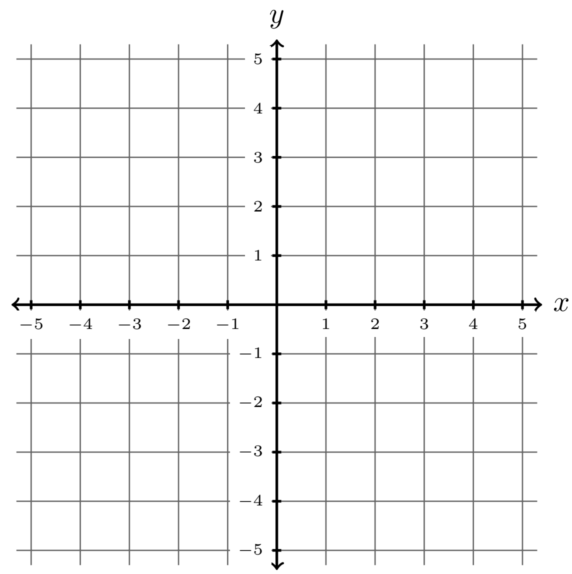
```
MaximumValue(A[1..n]):
  x = A[1]
  i = 2
  while i ≤ n do
      x = max(x, A[i])
      i = i + 1
  end while
  //x is now the maximum value in the array
  return x
```

8. [**5 points**] When x is a string over $\{a, b\}$, let x^I be the string obtained from x by "inverting" each letter: each a becomes b and each b becomes a. For example, if $x = abaa$, then $x^I = babb$. Give a recursive definition for x^I.

9. [**5 points**] Consider the following recursive definition of a set of points S in the plane with integer coordinates.

    1. $(0, 0) \in S$
    2. If $(x, y) \in S$, then $(x + 1, y + 1) \in S$.
    3. If $(x, y) \in S$, then $(x + 1, y - 1) \in S$.

    On the grid to the right, circle the points that are members of S.



10. [**5 points**] Solve the following recurrence: $T(1) = 1$, $T(2) = 1$, and $T(n) = 2T(n-1) + 8T(n-2)$ for $n \geq 3$.

11. [**6 parts, 1 point each**] Let $A = \{1, 2, \{2\}, \{4, 5, 6\}, 3\}$, $B = \{\{4, 5\}, 2\}$, and $C = \{\emptyset, \{2\}\}$.

    (a) True or false: $5 \in B$.

    (c) True or false: $\emptyset \subseteq B$.

    (b) True or false: $B \subseteq A$.

    (d) True or false: $\{2\} \subseteq C$.

    (e) Find $|A|$.

    (f) Find $A \cap C$. (Use set notation. Give an explicit list of the members.)

12. [**5 points**] Is $\{1, 2, 3, \ldots\} \times \{1, 2, 3, \ldots\}$ denumerable/countable? Explain why or why not.

13. [**5 points**] How many ways can you seat 3 men and 5 women in a row if all the men sit together and all the women sit together?

14. [**5 points**] The local supermarket has a sale on fruit; for a fixed price, you may choose 12 pieces of fruit from the supermarket's selection of apples, oranges, pears, and peaches. How many ways are there to select the fruit? For example, 10 apples, 1 orange, and 1 peach is one possible way.

15. [**2 parts, 3 points each**] In this problem, we explore a consequence of *Bayes' theorem.*

    (a) Prove that if E_1 and E_2 are events, then $P(E_1|E_2) = \dfrac{P(E_1)}{P(E_2)} \cdot P(E_2|E_1)$.
    [Hint: Start with the Right Hand Side. Apply the definition of conditional probability to $P(E_2|E_1)$.]

    (b) A new disease is discovered; experts estimate that 1% of the world's population is infected. A pharmaceutical company develops a test for the disease, and clinical trials are conducted. When administered to an infected person, the test gives a positive result 98% of the time. When administered to the general population, the test gives a positive result 3% of the time.

    A person named Sam is randomly selected from the world's population. Sam is tested for the disease and the test returns a positive result. What is the probability that Sam is actually infected?

    [Hint: Apply Bayes' theorem. What are E_1 and E_2?]

16. [**3 parts, 3 points each**] Finding coefficients. You do not need to simplify your answers to parts (b) and (c).

    (a) Find the numerical value of the coefficient of x^9y^3 in $(x+y)^{12}$.

    (b) Find the coefficient of $x^4y^4z^6$ in $(2x+3y+5z)^{14}$.

    (c) Find the coefficient of $w^5x^{10}y^3z^2$ in $(w+x+y+z)^{20}$.

17. [**5 points**] Use the Euclidian Algorithm to find and express $\gcd(14365, 5915)$ as a linear combination of 14365 and 5915.

18. [**3 points**] The first 50 primes are as follows: 2 3 5 7 11 13 17 19 23 29 31 37 41 43 47 53 59 61 67 71 73 79 83 89 97 101 103 107 109 113 127 131 137 139 149 151 157 163 167 173 179 181 191 193 197 199 211 223 227 229. Which of these primes do you need to check to determine whether 8521 is prime?

19. [**2 parts, 3 points each**] Let G be a connected simple planar graph with 57 vertices and 120 edges.

    (a) In a drawing of G, how many regions will there be?

    (b) Must G contain a triangle as a subgraph? Explain.

20. [**2 parts, 5 points each**] In the RSA algorithm, let $p = 59$ and $q = 83$. Then $n = 4897$ and $\varphi(n) = 4756$. For the encryption key, pick $e = 9$.

    (a) Use the Euclidean algorithm to find the decryption key d.

    (b) Encode $T = 3268$ using the public key (n, e).

21. Your research involves working with a long string consisting of the characters a, b, c, d, e, f, g, and h. You find that the characters occur with relative frequencies as follows.

a	b	c	d	e	f	g	h
11%	5%	2%	36%	3%	20%	8%	15%

(a) [**6 points**] Run the Huffman algorithm to produce a binary rooted tree.

(b) [**4 points**] Use the tree to give a Huffman code for this distribution of characters.

(c) [**2 points**] Find the average number of bits used to encode a character with this code.

22. [**5 points**] Recall that a *derangement* is a permutation that does not map any element to itself. For example, consider two permutations given by $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 4 & 3 & 1 \end{pmatrix}$ and $g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 2 & 3 & 1 \end{pmatrix}$. In this case, f is not a derangement because $f(2) = 2$, but g is a derangement.

    Find the number of permutations of $\{1, 2, 3, 4, 5\}$ that are derangements; do not simplify your answer for this part.

    [Hint: Use inclusion-exclusion to count the number of permutations that are not derangements.]

23. [**5 points**] Draw the expression tree for $[x + ((y - x) \times (z \div y))] - (x \div 3)$.