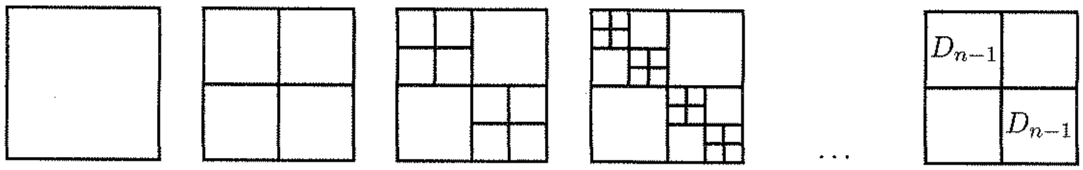
Math374 **Quiz #4** **13 October 2011**

Name: Solution

1. A sequence of geometric designs is defined recursively. The first design D_1 is a square. The nth design D_n is obtained by dividing a square into four quadrants. The upper left and lower right quadrants are scaled down copies of D_{n-1}.



D_1 D_2 D_3 D_4 ... D_n

(a) **[1 point]** The number of regions in D_1 is 1, in D_2 is 4, and in D_3 is 10. How many regions are in D_4, D_5, D_6, and D_7?

D_4: 22 D_5: 46 D_6: 94 D_7: 190

(b) **[1.5 points]** Define a recurrence relation $R(n)$ so that $R(n)$ is the number of regions in D_n.

$$R(n) = \begin{cases} 1 & n = 1. \\ 2R(n-1) + 2 & n \geq 2 \end{cases}$$

(c) **[1 bonus point]** Solve the recurrence you obtained in part (b).

$$R(n) = 2^{n-1} \cdot R(1) + \sum_{i=2}^{n} 2^{n-i} \cdot 2$$

$$= 2^{n-1} \cdot 1 + (2^{n-1} + 2^{n-2} + \dots + 2^1)$$

$$= 2^{n-1} + (2^{n-1} + 2^{n-2} + \dots + 2^1 + 1) - 1$$

$$= 2^{n-1} + \frac{2^n - 1}{2 - 1} - 1$$

$$= 2^{n-1} + 2^n - 1 - 1 = \boxed{2^n + 2^{n-1} - 2} = \boxed{3 \cdot 2^{n-1} - 2}$$

2. Let $A = \{\{3\}, 5, \{6\}, 8\}$, $B = \{\varnothing, \{3\}, 4, 5, 6\}$, and $C = \{\varnothing, \{4, 6\}\}$.

   (a) **[6 parts, 0.5 points each]** Which of the following statements are true?

   i. $\{4, 5, 6\} \subseteq B$ TRUE
   ii. $\{6\} \in A$ TRUE
   iii. $\varnothing \subseteq A$ TRUE
   iv. $3 \in A$ FALSE
   v. $\{4, 6\} \subseteq C$ FALSE
   vi. $\{4, 6\} \in C$ TRUE

   (b) **[0.5 points]** Find $B \cap C$.

   $$B \cap C = \{\varnothing\}$$

3. Let $T(n)$ be the following recurrence. $T(1) = 0$, $T(2) = 1$, and $T(n) = 10T(n-1) - 25T(n-2)$ for $n \geq 3$.

   (a) **[0.5 points]** Find the first five values of the sequence $T(n)$, from $T(1)$ to $T(5)$.

   $0, 1, 10, 75, 500$

   (b) **[1.5 points]** Solve the recurrence.

$$t^2 = 10t - 25$$
$$t^2 - 10t + 25 = 0$$
$$(t-5)^2 = 0$$
$$t = 5.$$

$\Rightarrow$ 1 root, so:

$$T(n) = p \cdot 5^{n-1} + q(n-1)5^{n-1}$$

Where

$\underline{T(1)}$: $0 = p$

$\underline{T(2)}$: $1 = p \cdot 5 + q \cdot 1 \cdot 5$

$1 = 0 + 5q$

$q = \frac{1}{5}$

$\underline{\text{So}}$: $T(n) = 0 \cdot 5^{n-1} + \frac{1}{5}(n-1) \cdot 5^{n-1}$

$= \boxed{(n-1) \cdot 5^{n-2}}$

4. **[2 parts, 1 point each]** An ATM pin number is a sequence of 4 digits.

   (a) How many pin numbers read the same forwards and backwards? For example, 2332 and 0000 count, but 9279 does not.

First ~~number~~ digit: 10 choices

Second digit: 10 choices

Third digit: (must equal 2^{nd}) 1 choice

4^{th} digit (must equal 1^{st}) 1 choice

TOTAL number $= 10 \cdot 10 \cdot 1 \cdot 1 = \boxed{100}$

   (b) How many pin numbers contain at least one 7? For example, 7284 and 4727 count, but 1234 does not.

- Total # pins = 10,000.
- Total # pins with no 7's:

1^{st} digit: 9 choices

2^{nd} digit 9 choices

3^{rd} digit 9 choices

4^{th} digit 9 choices

$9 \cdot 9 \cdot 9 \cdot 9 = (80+1)^2 = 6400 + 160 + 1 = 6561$

- # pins with a 7 $= 10{,}000 - 6{,}561 = \boxed{3439}$