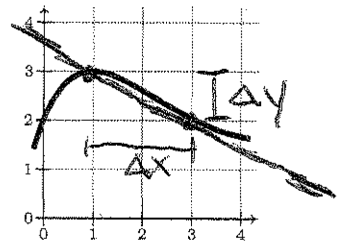
Name: Solutions

Directions: Show all work. No credit for answers without work.

1. [**2 parts, 4 points each**] The graph of $f(x)$ appears below.

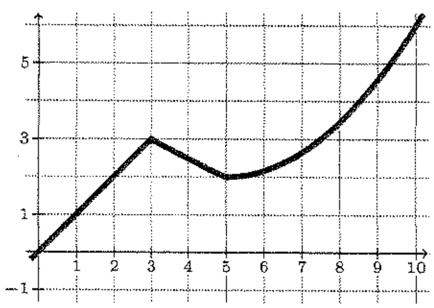


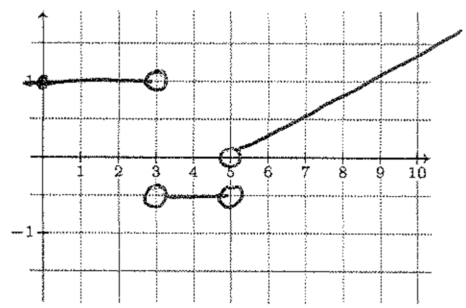
(a) Sketch the tangent line to $f(x)$ at $x = 3$ in the provided graph.

(b) Estimate $f'(3)$.

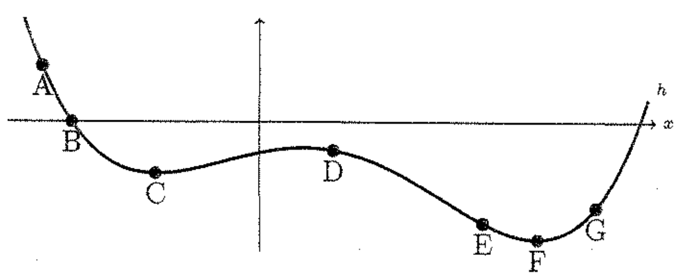
$$f'(3) \approx \frac{\Delta y}{\Delta x} = \boxed{\frac{-1}{2}}$$

2. [**6 points**] The graph of $g(x)$ appears below. Sketch $g'(x)$ in the space provided.





3. [**2 parts, 3 points each**] The following is a graph of $h(x)$. Some points are labeled.



(a) At which of the labeled points is the second derivative $h''(x)$ positive?

A, B, C, E, F, G

(b) At which of the labeled points is the second derivative $h''(x)$ negative?

D

4. **[4 parts, 2 points each]** A glass of water is removed from the refrigerator and placed on the counter. The temperature T of the water (in degrees Fahrenheit) is a function $T = f(x)$ of the time x (in minutes) since the water is exposed to room temperature.

(a) In $f(15) = A$, what are the units of 15? What are the units of A?

15: ~~[illegible]~~ $\boxed{\text{minutes}}$

A: $\boxed{{}^\circ F}$

(b) Do you expect the derivative f' to be positive or negative?

$f'(x) > 0$

positive

(c) In the statement $f'(15) = B$, what are the units of B?

B: $\boxed{{}^\circ F/\text{minute}}$

(d) Do you expect the second derivative f'' to be positive or negative?

$f''(x) < 0$

negative

5. **[2 parts, 3 points each]** Fill in the blanks. If $f''(x) > 0$, then

(a) $f'(x)$ is increasing, and

(b) $f(x)$ is concave up.

6. **[3 parts, 2 points each]** Let $C(q)$ be the cost (in dollars) of producing q items, and let $R(q)$ be the revenue (in dollars) received when producing q items.

(a) If $C(40) = 2320$ and $C'(40) = 15$, estimate $C(43)$.

$$C(43) \approx C(40) + \Delta q \cdot C'(40) = 2320 + 3 \cdot 15 = \boxed{\$2365}$$

(b) If $C'(40) = 15$ and $R'(40) = 18$, estimate the profit that results from producing the 41st item.

$$MP = MR - MC = 18 - 15 = \boxed{\$3}$$

(c) The current production level is 67 items, and $C(67) = 4208$, $C'(67) = 24$, $R(67) = 3100$, and $R'(67) = 32$. In these circumstances, should the company increase production or decrease production? Why?

$MR = 32$

$MC = 24$

Since $MR > MC$, the company should $\boxed{\text{increase}}$ production.

7. **[10 parts, 2 points each]** Differentiate the following functions.

(a) $y = 2x^8$

$$\boxed{16x^7}$$

(b) $y = \frac{4}{x^5} = 4x^{-5}$

$$\frac{d}{dx}\left[4x^{-5}\right] = 4\frac{d}{dx}\left[x^{-5}\right] = 4(-5)x^{-6} = \boxed{-20x^{-6}}$$

(c) $y = \sqrt{x} = x^{\frac{1}{2}}$

$$\frac{d}{dx}\left[x^{\frac{1}{2}}\right] = \frac{1}{2}x^{-\frac{1}{2}} = \boxed{\frac{1}{2\sqrt{x}}}$$

(d) $y = 3x^7 - x^2$

$$\frac{d}{dx}\left[3x^7 - x^2\right] = 3\frac{d}{dx}\left[x^7\right] - \frac{d}{dx}\left[x^2\right] = 3\cdot 7x^6 - 2x = \boxed{21x^6 - 2x}$$

(e) $y = e^{-x}$

$$\frac{d}{dx}\left[e^{-1\cdot x}\right] = (-1)\cdot e^{-1\cdot x} = \boxed{-e^{-x}}$$

(f) $y = 4^x$

$$\boxed{(\ln 4)\cdot 4^x}$$

(g) $y = e^x + x^e$

$$\frac{d}{dx}\left[e^x + x^e\right] = \frac{d}{dx}\left[e^x\right] + \frac{d}{dx}\left[x^e\right] = \boxed{e^x + e\cdot x^{e-1}}$$

(h) $y = \ln(x)$

$$\frac{d}{dx}\left[\ln(x)\right] = \boxed{\frac{1}{x}}$$

(i) $y = 2(1.09)^x + x^{1.2} + \ln\left(\sqrt{5}\right)$

const

$$\frac{dy}{dx} = 2\frac{d}{dx}\left[(1.09)^x\right] + \frac{d}{dx}\left[x^{1.2}\right] + \frac{d}{dx}\left[\ln(\sqrt{5})\right] = \boxed{2\ln(1.09)(1.09)^x + 1.2x^{0.2}} + 0$$

(j) $y = \dfrac{e^3 - \ln(\ln(4.26))}{2\pi + \sqrt{11}\ln(3)}$

const

$$\boxed{0}$$

8. [**4 parts, 5 points each**] Differentiate the following functions.

(a) $y = (x^2 + 6x + 1)^{15}$

$$\frac{d}{dx}\left[(x^2+6x+1)^{15}\right] = 15(x^2+6x+1)^{14} \cdot \frac{d}{dx}\left[x^2+6x+1\right]$$

$$= \boxed{15(x^2+6x+1)^{14}(2x+6)}$$

(b) $y = \dfrac{x^3 - x^2}{e^x + 5}$

$$\frac{dy}{dx} = \frac{(e^x+5)\frac{d}{dx}\left[x^3-x^2\right] - (x^3-x^2)\frac{d}{dx}\left[e^x+5\right]}{(e^x+5)^2}$$

$$= \boxed{\frac{(e^x+5)(3x^2-2x) - (x^3-x^2)\cdot e^x}{(e^x+5)^2}}$$

(c) $y = x^2 e^{7x}$

$$\frac{dy}{dx} = \frac{d}{dx}\left[x^2\right]\cdot e^{7x} + x^2 \cdot \frac{d}{dx}\left[e^{7x}\right]$$

$$= 2xe^{7x} + x^2 \cdot 7e^{7x} = \boxed{xe^{7x}(2+7x)}$$

(d) $y = \ln(x \ln(x))$

$$\frac{d}{dx}\left[\ln(x\ln(x))\right] = \frac{1}{x\ln(x)} \cdot \frac{d}{dx}\left[x\ln(x)\right]$$

$$= \frac{1}{x\ln(x)} \cdot \left(\frac{d}{dx}[x]\cdot \ln(x) + x \cdot \frac{d}{dx}\left[\ln(x)\right]\right)$$

$$= \frac{1}{x\ln(x)} \cdot \left(\ln(x) + x \cdot \frac{1}{x}\right)$$

$$= \frac{1}{x\ln(x)} \cdot (\ln(x)+1) = \boxed{\frac{\ln(x)+1}{x\ln(x)}}$$

9. [**8 points**] Find the equation of the line tangent to the function $f(x) = (2x-1)^3$ at $x = 2$.

Point:

$x_0 = 2$

$y_0 = f(2) = (2\cdot 2 - 1)^3$
$= 3^3 = 27$

Slope:

$f'(x) = \frac{d}{dx}\left[(2x-1)^3\right]$
$= 3(2x-1)^2 \cdot \frac{d}{dx}[2x-1]$
$= 6(2x-1)^2$

$m = f'(2) = 6\cdot(2\cdot 2 - 1)^2$
$= 6 \cdot 3^2 = 54$

$y - y_0 = m(x - x_0)$

$\boxed{y - 27 = 54(x-2)}$ or

$\boxed{y = 54x - 81}$

10. Let $g(x) = (x-4)^3(2x+1)^2$.

(a) [**6 points**] Find $g'(x)$ in factored form.

$g'(x) = \frac{d}{dx}\left[(x-4)^3 \cdot (2x+1)^2\right]$

$= \frac{d}{dx}\left[(x-4)^3\right]\cdot(2x+1)^2 + (x-4)^3 \frac{d}{dx}\left[(2x+1)^2\right]$

$= 3(x-4)^2 \cdot \frac{d}{dx}[x-4] \cdot (2x+1)^2 + (x-4)^3 \cdot 2(2x+1)\cdot \frac{d}{dx}[2x+1]$

$= 3(x-4)^2 \cdot 1 \cdot (2x+1)^2 + (x-4)^3 \cdot 2(2x+1) \cdot 2$

factor: $= (x-4)^2(2x+1)\left[3(2x+1) + 4(x-4)\right]$

$= (x-4)^2(2x+1)(6x+3+4x-16) = (x-4)^2(2x+1)(10x-13)$

(b) [**6 points**] Make a sign chart for $g'(x)$ and classify each critical point of $g(x)$ as a local minimum, a local maximum, or neither.

Critical pts:

$(x-4)^2 = 0$ or $2x+1 = 0$ or $10x - 13 = 0$

$x - 4 = 0$
$x = 4$

$x = -\frac{1}{2}$

$x = \frac{13}{10} = 1.3$

- $g'(-1) =$ pos $\cdot$ neg $\cdot$ neg $=$ pos
- $g'(0) =$ pos $\cdot$ pos $\cdot$ neg $=$ neg
- $g'(2) =$ pos $\cdot$ pos $\cdot$ pos $=$ pos
- $g'(5) =$ pos $\cdot$ pos $\cdot$ pos $=$ pos

Sign Chart: