Announcements

- Email me by 5:00pm if you are not willing to take Test #2 on Wed Oct 20 OR are not willing to accept the delayed return of your tests (until Nov 10)
- Test #2: officially scheduled for Mon Oct 18
- Quiz #6 due tonight 11pm
- HW #7 out; due Wed Oct 13.

Last time:

Ex Find and classify the critical points of $f(x) = x^3 - 3x^2 + 3x + 2$.

Soln: Step 1: Find critical points:

- $f'(x) = \frac{d}{dx}[f(x)] = \frac{d}{dx}\left[x^3 - 3x^2 + 3x + 2\right]$

$$= 3x^2 - 6x + 3$$

- Solve for x in $3x^2 - 6x + 3 = 0$

$$3(x^2 - 2x + 1) = 0$$

$$3(x - 1)(x - 1) = 0$$

$$x - 1 = 0 \quad \text{or} \quad x - 1 = 0$$

$$x = 1 \qquad\qquad x = 1$$

- So there is one critical point of f at $x = 1$.

Step 2: Classify the critical points

- $f''(x) = \frac{d}{dx}[f'(x)] = \frac{d}{dx}\left[3x^2 - 6x + 3\right]$

$$= 6x - 6$$

- $f''(1) = 6(1) - 6 = 0$ // $\Rightarrow$ SDT is inconclusive

Need to apply F.D.T:

- $f(x) = x^3 - 3x^2 + 3x + 2$
- $f'(x) = 3x^2 - 6x + 3$

$$= 3(x-1)(x-1)$$
$$= 3(x-1)^2$$

Sign chart for f':

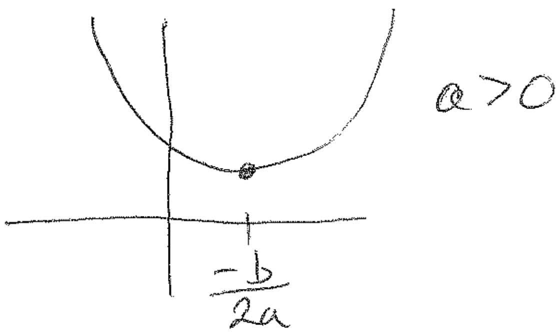
f:	$\nearrow$		$\nearrow$	
$f'(x)$:	$+$	0	$+$	
x	0	1		

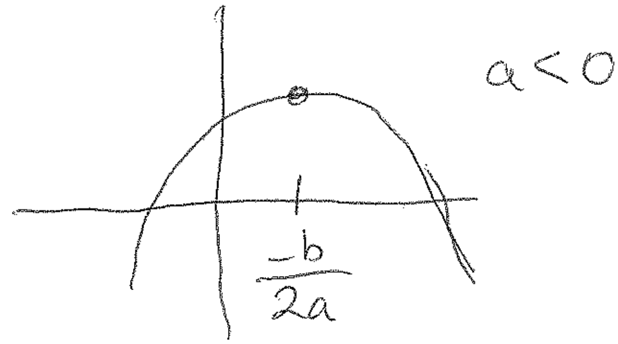
critical point

FDT: The sign of f' does not change as x passes through 1, so 1 is neither a local min nor a local max.

• **Ex:** Let $a, b,$ and c be constants with $a \neq 0$.

Find the vertex of the parabola $f(x) = ax^2 + bx + c$.





Soln: Find Local min / Local max:

$$f'(x) = \frac{d}{dx}\left[ax^2 + bx + c\right]$$

$$= \frac{d}{dx}\left[ax^2\right] + \frac{d}{dx}\left[bx\right] + \frac{d}{dx}\left[c\right]$$

$$= a\frac{d}{dx}\left[x^2\right] + b\frac{d}{dx}\left[x\right] + c\frac{d}{dx}\left[1\right]$$

$$= a(2x) + b \cdot 1 + c \cdot 0$$

$$= 2ax + b$$

$\underline{f'(x) = 0}$: Solve for x in

$$2ax + b = 0$$

$$2ax = -b$$

$$x = \frac{-b}{2a}$$

• $f\left(\frac{-b}{2a}\right) = a\left(\frac{-b}{2a}\right)^2 + b \cdot \left(-\frac{b}{2a}\right) + c$

$$= a\left(\frac{b^2}{4a^2}\right) - \frac{b^2}{2a} + c$$

$$= \frac{b^2}{4a} - \frac{b^2}{2a} + c$$

$$= \frac{b^2}{4a} - \frac{2b^2}{4a} + c$$

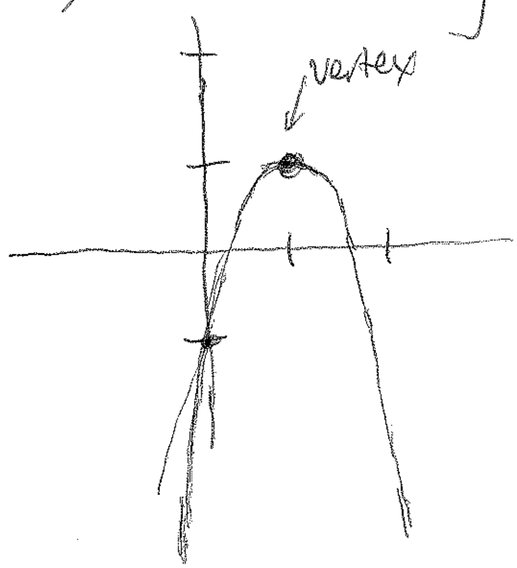
$$= \frac{b^2 - 2b^2}{4a} + c$$

$$= \frac{-b^2}{4a} + c$$

- So the vertex is at $\boxed{\left(\frac{-b}{2a}, \frac{-b^2}{4a} + c\right)}$.

Ex: Find the equation of a parabola with vertex $(1,1)$ containing the point $(0,-1)$.

Soln



- $f(x) = ax^2 + bx + c$

Know: $f(0) = -1$

$$a \cdot 0^2 + b \cdot 0 + c = -1$$

$$\boxed{c = -1}.$$

- The parabola $f(x) = ax^2 + bx + c$ has its vertex at $x = \frac{-b}{2a}$ and $y = \frac{-b^2}{4a} + c$.

• Want: $\quad 1 = \frac{-b}{2a} \quad$ and $\quad 1 = \frac{-b^2}{4a} + (-1)$

$$2 = \frac{-b^2}{4a}$$

$$2a = -b \qquad\qquad 2a = \frac{-b^2}{4}$$

• So: $\quad -b = \frac{-b^2}{4}$

$$b = \frac{b^2}{4}$$

$$4b = b^2$$

$$0 = b^2 - 4b \qquad\qquad \boxed{c = -1}$$

$$0 = b(b-4)$$

$$b = 0 \quad \text{or} \quad b = 4$$

• So $\quad a = \frac{-b}{2}$; $\quad \cancel{\boxed{\begin{matrix} b = 0 \\ a = 0 \end{matrix}}}$ or $\boxed{\begin{matrix} b = 4 \\ a = -\frac{4}{2} = -2 \end{matrix}}$

not a parabola $\qquad\qquad$ *

• Soln: $\quad f(x) = ax^2 + bx + c$

$$= \boxed{-2x^2 + 4x - 1}$$

4.3: Global Maxima and Minima:

- A function f has a global minimum at p if f(p) is ~~at least~~ less than or equal to ~~as large as~~ all values of f.

- A function f has a global maximum at p if f(p) is greater than or equal to all values of f.

- To Find Global Maxima/Minima of a continuous function f over [a, b]:

  1. Find the critical points of f (inside [a,b])

  2. Evaluate f ~~at~~ at critical points and the endpoints (a, and b).

  3. Largest value(s) $\Longrightarrow$ Global maximums

Smallest value(s) $\Rightarrow$ Global minimums

Ex: Find the global max and min of the function $f(x) = \dfrac{x}{1+x^2}$ on $[0,3]$.

Soln: Find critical points:

- $f'(x) = \dfrac{d}{dx}\left[\dfrac{x}{1+x^2}\right]$

$$= \frac{(1+x^2)\frac{d}{dx}[x] - x\frac{d}{dx}[1+x^2]}{(1+x^2)^2}$$

$$= \frac{(1+x^2)\cdot 1 - x(2x)}{(1+x^2)^2}$$

Solve for x:

$$\cancel{(1+x^2)^2} \cdot \frac{(1+x^2) - 2x^2}{\cancel{(1+x^2)^2}} = 0 \cdot (1+x^2)^2$$

$$1 + x^2 - 2x^2 = 0$$

$$1 - x^2 = 0$$

$$1 = x^2$$

$$x = \pm 1.$$

Critical points inside $[0,3]$: $+1$.

$$f(x) = \frac{x}{1+x^2}$$

2. smallest $\Rightarrow f(0) = 0$

largest $\Rightarrow f(1) = \frac{1}{1+1^2} = \frac{1}{2} = 0.5$

$f(3) = \frac{3}{1+3^2} = \frac{3}{10} = 0.3$

So global maximum at $x = 1$

global minimum at $x = 0$.

CLASS 01/31 == 1:25 pm – 2:15 pm

Ex: Find and Classify (with the FDT) the critical points of $f(x) = 2x^3 - 5x^2 - 4x + 1$.

Soln: Find the critical points:

- $$f'(x) = \frac{d}{dx}\left[2x^3 - 5x^2 - 4x + 1\right]$$

$$= 6x^2 - 10x - 4$$

$$= 2(3x^2 - 5x - 2)$$

$$= 2(3x + 1)(x - 2)$$

Solve for x: $2(3x+1)(x-2) = 0$

$3x + 1 = 0$ or $x - 2 = 0$

$3x = -1$ $\quad$ $x = 2$

$x = -\frac{1}{3}$

- Critical points: $-\frac{1}{3}, 2$

Classify $-\frac{1}{3}, 2$ as local min/max or neither

Using FDT:

f:	↗		↘		↗
$f'(x)$	$+$	0	$-$	0	$+$
x	-1	$-\frac{1}{3}$	0	2	3

$-\frac{1}{3}$: Local max

2: Local min

Classify using SDT: $-\frac{1}{3}, 2$.

$$f''(x) = \frac{d}{dx}\left[6x^2 - 10x - 4\right]$$

$$= 12x - 10$$

$$f''\left(-\tfrac{1}{3}\right) = 12\left(-\tfrac{1}{3}\right) - 10 = -4 - 10 = -14$$

 $\longrightarrow$ Local max

$f''(2) = 12(2) - 10 = 14$

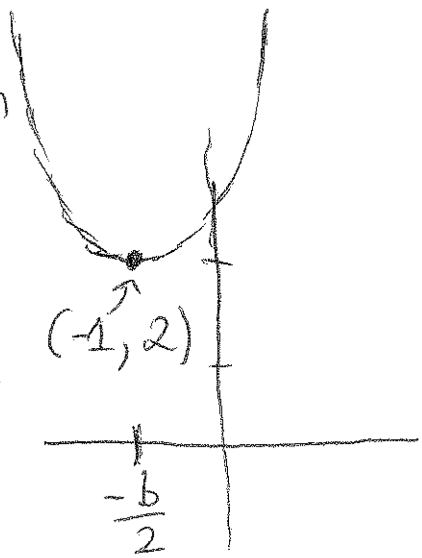
$\Longrightarrow$ Local min.

Ex: Find constants b and c ~~such the~~ that make

$f(x) = x^2 + bx + C$ a parabola whose vertex

is at $(-1, 2)$.

Soln



- The vertex $(-1, 2)$ is a local minimum!

- Find the local min. of $f(x) = x^2 + bx + c$.

- $f'(x) = \frac{d}{dx}\left[x^2 + bx + c\right]$

$$= \frac{d}{dx}\left[x^2\right] + \frac{d}{dx}\left[bx\right] + \frac{d}{dx}\left[c\right]$$

$$= 2x + b + 0$$

- Solve for x in $2x + b = 0$

$$2x = -b$$
$$x = \frac{-b}{2}$$

- Know that $x = \frac{-b}{2}$ is the location of the local min.

- Want $\frac{-b}{2} = -1$.
$$-b = -2$$
$$b = 2$$

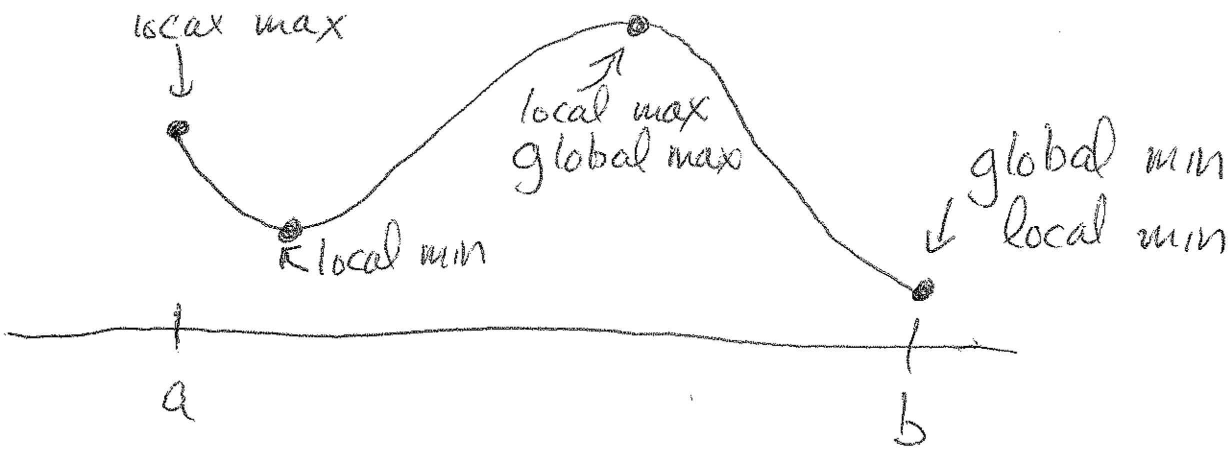
Know: $f(x) = x^2 + 2x + c$

- $f(-1) = 2$
- $(-1)^2 + 2(-1) + c = 2$
$$1 - 2 + c = 2$$
$$-1 + c = 2$$
$$c = 3$$

Soln: $\boxed{f(x) = x^2 + 2x + 3}$.

4.3 Global Minima/Maxima

- f has a global minimum at p if $f(p)$ is less than or equal to all values of f
- f has a global maximum at p if $f(p)$ is greater than or equal to all values of f.



- To Find Global Minima/Maxima of a continuous function f on $[a,b]$:

  1. Find the critical points of f inside $[a,b]$
  2. Evaluate f at critical points and the endpoints (a and b).
  3. Smallest value(s) $\Longrightarrow$ Global min's

     Largest value(s) $\Longrightarrow$ Global max's

Ex: Find the global maximum value and global minimum value of $f(x) = \frac{x}{1+x^2}$ over $[0,3]$.

Soln: Find critical points.

- $f'(x) = \frac{d}{dx}\left[\frac{x}{1+x^2}\right]$

$$= \frac{(1+x^2)\frac{d}{dx}[x] - x\frac{d}{dx}[1+x^2]}{(1+x^2)^2}$$

$$= \frac{(1+x^2)\cdot 1 - x(2x)}{(1+x^2)^2}$$

$$= \frac{1+x^2 - 2x^2}{(1+x^2)^2}$$

$$= \frac{1-x^2}{(1+x^2)^2}$$

- Solve for x in $\frac{1-x^2}{(1+x^2)^2} = 0$

$$\cancel{(1+x^2)^2}\cdot\frac{1-x^2}{\cancel{(1+x^2)^2}} = 0\cdot(1+x^2)^2$$

$$1-x^2 = 0$$

$$1 = x^2$$

$$x^2 = 1$$

$$x = \pm 1$$

$x^2 - 1 = 0$

$(x+1)(x-1) = 0$

$x = -1$ or $x = 1$

- Critical points in $[0,3]$: 1 $\qquad f(x) = \dfrac{x}{1+x^2}$

- $f(1) = \dfrac{1}{1+1^2} = \dfrac{1}{2} = 0.5$

  $f(0) = 0 = 0$

  $f(3) = \dfrac{3}{1+3^2} = \dfrac{3}{10} = 0.3$

- Global min occurs at $x=0$ with value 0.

  Global max occurs at $x=1$ with value 0.5.