 Example like # 27 : $(15)^{1/3} = \sqrt[3]{15}$

$\boxed{(a)}$ Use a Taylor polynomial of degree 4 to approximate the given number.

Soln:

- Use $f(x) = x^{1/3}$, Taylor series centered at $c = 8$.
  ($c = 27$ will also work)

- $$f(x) = \sum_{k=0}^{\infty} a_k (x-c)^k = \sum_{k=0}^{\infty} a_k (x-8)^k$$

- $$a_k = \frac{f^{(k)}(c)}{k!} = \frac{f^{(k)}(8)}{k!}$$

- Find Derivatives of $f(x)$:

$$\Rightarrow f^{(0)}(x) = x^{1/3}$$

$$\Rightarrow f^{(1)}(x) = \frac{1}{3} x^{\left(\frac{1}{3}-1\right)}$$

$$\Rightarrow f^{(2)}(x) = \frac{1}{3}\left(\frac{1}{3}-1\right) x^{\left(\frac{1}{3}-2\right)}$$

$$\Rightarrow f^{(3)}(x) = \frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right) x^{\left(\frac{1}{3}-3\right)}$$

$$\vdots$$

$$(k \geq 1) \Rightarrow f^{(k)}(x) = \frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)\cdots\left(\frac{1}{3}-(k-1)\right) x^{\left(\frac{1}{3}-k\right)}$$

$(k\geq 1)$ • So, $f^{(k)}(c) = f^{(k)}(8) = \frac{1}{3}(\frac{1}{3}-1)(\frac{1}{3}-2)\cdots(\frac{1}{3}-(k-1))\, 8^{1/3-k}$

$$= \frac{1}{3}(\frac{1}{3}-1)(\frac{1}{3}-2)\cdots(\frac{1}{3}-(k-1)) \cdot \frac{8^{1/3}}{8^k}$$

$$= \frac{2}{3\cdot 8^k}\underbrace{(\frac{1}{3}-1)(\frac{1}{3}-2)\cdots(\frac{1}{3}-(k-1))}_{\text{note: } k-1 \text{ factors}}$$

Find Taylor polynomial of degree 4:

- $a_0 = \dfrac{f^{(0)}(8)}{0!} = \dfrac{(8)^{1/3}}{1} = 2$

- $a_1 = \dfrac{f^{(1)}(8)}{1!} = \dfrac{2}{3\cdot 8} = \dfrac{1}{12}$

- $a_2 = \dfrac{f^{(2)}(8)}{2!} = \dfrac{1}{2!}\left(\dfrac{2}{3\cdot 8^2}(\frac{1}{3}-1)\right) = \dfrac{1}{3\cdot 8^2}\left(-\dfrac{2}{3}\right) = -\dfrac{1}{9\cdot 32} = -\dfrac{1}{288}$

- $a_3 = \dfrac{f^{(3)}(8)}{3!} = \dfrac{1}{3!}\left(\dfrac{2}{3\cdot 8^3}(\frac{1}{3}-1)(\frac{1}{3}-2)\right) = \dfrac{5}{20736} \approx 2.41\cdot 10^{-4}$

- $a_4 = \dfrac{f^{(4)}(8)}{4!} = \dfrac{1}{4!}\left(\dfrac{2}{3\cdot 8^4}(\frac{1}{3}-1)(\frac{1}{3}-2)(\frac{1}{3}-3)\right) = \dfrac{-5}{248{,}832} \approx -2.0094\cdot 10^{-5}$

- $P_4(x) = a_0 + a_1(x-8) + a_2(x-8)^2 + a_3(x-8)^3 + a_4(x-8)^4$
- $P_4(x) = 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2 + \frac{5}{20736}(x-8)^3 - \frac{5}{248{,}832}(x-8)^4$

- $P_4(15) = 2 + \frac{1}{12}\cdot 7 - \frac{1}{288}(7)^2 + \frac{5}{20736}(7)^3 - \frac{5}{248{,}832}(7)^4$

$$= \frac{609055}{248832} \approx \boxed{2.448}$$

$\boxed{(b)}$ ~~Estimate~~ Bound the error in this approximation.

Soln: • $P(15) = 2 + \frac{7}{12} - \frac{(7)^2}{288} + \frac{5(7)^3}{20736} - \frac{5(7)^4}{248{,}832} + b_5 - b_6 \cdots$

$\approx (2.58) - 0.17 + 0.083 - 0.048 + b_5 - b_6 \cdots$

Note: $P(15)$ is an alternating series, whose terms are decreasing in magnitude:

$$2.58 > 0.17 > 0.083 > 0.048 > \cdots$$

(How could you verify this holds in general, for all terms?)

Therefore the error is bounded by the magnitude of the next term:

$$\left| (15)^{1/3} - P_4(15) \right| \leq |b_5|.$$

• We need to find one more term in this series:

• $$a_5 = \frac{f^{(5)}(8)}{5!} = \frac{1}{5!}\left(\frac{2}{3 \cdot 8^5} \left(\tfrac{1}{3}-1\right)\left(\tfrac{1}{3}-2\right)\left(\tfrac{1}{3}-3\right)\left(\tfrac{1}{3}-4\right) \right)$$

$$= \frac{11}{5{,}971{,}968} \approx 1.84 \cdot 10^{-6}$$

• $$b_5 = a_5 (x-8)^5 = a_5 \cdot 7^5 = \frac{184{,}877}{5{,}971{,}968} \approx 0.031$$

• Therefore $\boxed{\left|(15)^{1/3} - P_4(15)\right| \le 0.031}$, ~~so $(15)^{1/3} = 2.448 \pm 0.031$~~

so

$$P_4(15) - 0.031 \le (15)^{1/3} \le P_4(15) + 0.031$$

$$\boxed{2.417 \le (15)^{1/3} \le 2.479}$$

(Actually, because the next term b_5 in the alternating series is positive, we know that

$$P_4(15) \le (15)^{1/3} \le P_4(15) + 0.031$$

$$2.448 \le (15)^{1/3} \le 2.479. \quad)$$

$\boxed{\text{(C)}}$ Bound the number of terms needed in a Taylor polynomial to guarantee an accuracy of 10^{-10}.

• Solution 1: Alternating series:
Must find n large enough so that $|b_n| < 10^{-10}$, and then the series $(b_0 + b_1) - \cdots + b_{n-1}$ has enough terms.

$$|b_n| = |a_n(x-8)^n| = |a_n \cdot 7^n|$$

$$= \left| \frac{1}{n!} \cdot \frac{2}{3 \cdot 8^n} \left(\tfrac{1}{3}-1\right) \cdots \left(\tfrac{1}{3}-(n-1)\right) \cdot 7^n \right|$$

$$= \frac{2}{3} \cdot \left(\frac{7}{8}\right)^n \cdot \frac{1}{n!} \cdot \left| \left(\tfrac{1}{3}-1\right)\left(\tfrac{1}{3}-2\right) \cdots \left(\tfrac{1}{3}-(n-1)\right) \right|$$

$(n \geq 2)$

$$= \frac{2}{3} \cdot \left(\frac{7}{8}\right)^n \cdot \frac{1}{n!} \left| \left(-\tfrac{2}{3}\right)\left(-\tfrac{5}{3}\right)\left(-\tfrac{8}{3}\right) \cdots \left(\tfrac{4-3n}{3}\right) \right|$$

$$= \frac{2}{3}\left(\frac{7}{8}\right)^n \cdot \frac{1}{n!} \left(\frac{2}{3} \cdot \frac{5}{3} \cdot \frac{8}{3} \cdots \frac{3n-4}{3} \right)$$

$$\leq \frac{2}{3}\left(\frac{7}{8}\right)^n \cdot \frac{1}{n!} \left(1 \cdot 2 \cdot 3 \cdots n-1\right)$$

$$\leq \frac{2}{3}\left(\frac{7}{8}\right)^n \cdot \frac{(n-1)!}{n!}$$

$$\leq \frac{2}{3}\left(\frac{7}{8}\right)^n \cdot \frac{1}{n}$$

· So we want n large enough so that

$$\frac{2}{3}\left(\frac{7}{8}\right)^n \cdot \frac{1}{n} < 10^{-10}$$

$$\left(\frac{7}{8}\right)^n < \frac{3}{2} n \cdot 10^{-10}$$

$$7^n \cdot 10^{10} < \frac{3}{2} n \cdot 8^n$$

Take log: $\quad n \ln 7 + 10 \ln 10 < n \ln 8 + \ln\left(\frac{3}{2}\right) + \ln(n)$

$$10\ln(10) - \ln\left(\tfrac{3}{2}\right) < n(\ln 8 - \ln 7) + \ln(n)$$

$$22.62 < n(0.1335) + \ln(n) \qquad (*)$$

- **Bound**: n at least $\frac{22.62}{0.1335} \approx 169.44$ will work.

- Smallest n which makes $(*)$ true is $n = 133$.

- So, using a Taylor polynomial of degree $\boxed{132}$ will give $(15)^{1/3}$ to within 10^{-10}.

- Using the bound from our guess (Taylor polynomial of degree $\boxed{169}$) is also a reasonable answer.

Note: It is also possible to solve this problem using Taylor's Theorem

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}, \quad \text{for some } z \text{ between } x \text{ and } c.$$

Solution 2: Taylor's Theorem:

- Want n large enough so that $|R_n(x)| < 10^{-10}$.

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}$$

$$= \frac{f^{(n+1)}(z)}{(n+1)!}(15-8)^{n+1}$$

$$= \frac{1}{(n+1)!} \cdot \underbrace{\left(\tfrac{1}{3}\right)\cdot\left(\tfrac{1}{3}-1\right)\left(\tfrac{1}{3}-2\right)\cdots\left(\tfrac{1}{3}-n\right)\cdot z^{1/3-(n+1)}} \cdot 7^{n+1}$$

(see bottom of p.1 for the formula for $f^{(n+1)}(z)$)

where z is between $c=8$ and $x=15$. For positive z, the function $z^{1/3-(n+1)}$ is decreasing (why? what is the sign of the derivative?), so $|z^{1/3-(n+1)}|$ is maximized when $z=8$. Thus:

$$|R_n(x)| \le \left|\frac{1}{(n+1)!}\left(\tfrac{1}{3}\right)\cdot\left(\tfrac{1}{3}-1\right)\left(\tfrac{1}{3}-2\right)\cdots\left(\tfrac{1}{3}-n\right)\cdot\frac{8^{1/3}}{8^{(n+1)}}\cdot 7^{(n+1)}\right|$$

$$= \frac{1}{(n+1)!}\,\frac{1}{3}\cdot\underbrace{\frac{2}{3}\cdot\cdots\cdot\frac{3n-1}{3}}_{n!}\cdot 2\cdot\left(\frac{7}{8}\right)^{n+1}$$

$$\le \frac{1}{(n+1)!}\cdot\frac{1}{3}\quad n!\quad\cdot 2\cdot\left(\frac{7}{8}\right)^{n+1} = \frac{2}{3n}\left(\frac{7}{8}\right)^{n+1}.$$

- So, if $\frac{2}{3n}\left(\frac{7}{8}\right)^{n+1} < 10^{-10}$, then the Taylor polynomial of degree n is accurate enough.

- (Almost) Same inequality as in solution 1; $n = 132$ is large enough.

- So Taylor's theorem also ~~says~~ implies that a Taylor polynomial of degree $\boxed{132}$ is accurate enough.