6.4 #24,

$$\frac{4}{x^3-2x^2+4x} = \frac{4}{x(x^2-2x+4)}$$

$$\overset{(\text{WANT})}{=} \frac{A}{x} + \frac{Bx+C}{x^2-2x+4}$$

Because x^2-2x+4 does not factor into linear terms, we need to look for a partial fraction decomposition with $Bx+C$ on top.

$$= \frac{A(x^2-2x+4)+(Bx+C)x}{x(x^2-2x+4)}$$

$$= \frac{Ax^2-2Ax+4A+Bx^2+Cx}{x(x^2-2x+4)}$$

(collect terms) $$= \frac{(A+B)x^2+(C-2A)x+4A}{x(x^2-2x+4)}$$

- So, we want $(A+B)x^2+(C-2A)x+4A = 4$
$$= 0x^2+0x+4.$$

- Set coefficients of like powers equal:

(x^2): $A+B=0$

(x^1): $C-2A=0$

(x^0): $4A=4$

- Now solve for A, B, C:

- $4A = 4$, so $\boxed{A = 1}$.

- $A + B = 0$

  $1 + B = 0$, so $\boxed{B = -1}$

- $C - 2A = 0$

  $C - 2 = 0$, so $\boxed{C = 2}$.

(To check our work, we can always simplify

$$\frac{1}{x} + \frac{-x+2}{x^2-2x+4}$$

and check that this equals $\frac{4}{x^3-2x^2+4x}$ ← what we started with.)

So: $\displaystyle\int \frac{4}{x^3-2x^2+4x}\,dx = \int \frac{1}{x} + \frac{-x+2}{x^2-2x+4}\,dx$

$$= \int \frac{1}{x}\,dx + \int \frac{-x+2}{x^2-2x+4}\,dx$$

$$= \ln|x| + \int \frac{-x+2}{x^2-2x+4}\,dx$$

- To solve $\displaystyle\int \frac{-x+2}{x^2-2x+4}\,dx$, we want to use a substitution:

$$u = x^2-2x+4$$
$$du = (2x-2)\,dx$$

- How can we change $-x+2$, into $du = (2x-2)dx$ in $\int \frac{-x+2}{x^2-2x+4}\,dx$?

- First, change the coefficient on x so that $-x$ becomes $2x$!

$$\int \frac{-x+2}{x^2-2x+4}\,dx = \int \frac{-2}{-2}\cdot\frac{-x+2}{x^2-2x+4}\,dx$$

$$= -\frac{1}{2}\int \frac{2x-4}{x^2-2x+4}\,dx$$

- Now, split the -4 into -2 for our copy of du and whatever is left:

$$-\frac{1}{2}\int \frac{2x-4}{x^2-2x+4}\,dx = -\frac{1}{2}\int \frac{(2x-2)-2}{x^2-2x+4}\,dx$$

$$= -\frac{1}{2}\int \frac{2x-2}{x^2-2x+4} - \frac{2}{x^2-2x+4}\,dx$$

$$= -\frac{1}{2}\left(\int \frac{2x-2}{x^2-2x+4}\,dx - \int \frac{2}{x^2-2x+4}\,dx\right)$$

$$= -\frac{1}{2}\left(\int \frac{1}{u}\,du - \int \frac{2}{x^2-2x+4}\,dx\right)$$

$$= -\frac{1}{2}\left(\ln|u| - 2\int \frac{1}{x^2-2x+4}\,dx\right)$$

$$= -\frac{1}{2}\ln|x^2-2x+4| + \int \frac{1}{x^2-2x+4}\,dx$$

- OK, let's review our progress:

$$\int \frac{4}{x^3-2x^2+4x}\,dx = \ln|x| + \int \frac{-x+2}{x^2-2x+4}\,dx$$
$$= \ln|x| - \frac{1}{2}\ln|x^2-2x+4| + \int \frac{1}{x^2-2x+4}\,dx.$$

- To solve $\int \frac{1}{x^2-2x+4}\,dx$, we want to transform it into an integral like $\int \frac{1}{u^2+1}\,du = \tan^{-1}(u) + C$.

- First, we complete the square:

$$x^2-2x+4 = (x-1)^2+3.$$

- Next, factor out a 3 so that we get a +1 const. term:

$$= 3\left(\frac{(x-1)^2}{3}+1\right).$$
$$= 3\left(\left(\frac{x-1}{\sqrt{3}}\right)^2+1\right).$$

- So, $\int \frac{1}{x^2-2x+4}\,dx = \int \frac{1}{3\left(\left(\frac{x-1}{\sqrt{3}}\right)^2+1\right)}\,dx$

$$= \frac{1}{3}\int \frac{1}{\left(\frac{x-1}{\sqrt{3}}\right)^2+1}\,dx$$

- Now, we are ready for a substitution: $u = \frac{x-1}{\sqrt{3}}$

$du = \frac{1}{\sqrt{3}}\,dx$

- $$\frac{1}{3}\int \frac{1}{\left(\frac{x-1}{\sqrt{3}}\right)^2+1}\,dx = \frac{1}{3}\int \frac{1}{u^2+1}\cdot\left(\sqrt{3}\,du\right)$$

$$= \frac{\sqrt{3}}{3}\int \frac{1}{u^2+1}\,du$$

$$= \frac{\sqrt{3}}{3}\tan^{-1}(u) + C$$

$$= \frac{\sqrt{3}}{3}\tan^{-1}\left(\frac{x-1}{\sqrt{3}}\right) + C$$

- So, our final answer is

$$\int \frac{4}{x^3-2x^2+4x}\,dx = \boxed{\ln|x| - \frac{1}{2}\ln|x^2-2x+4| + \frac{\sqrt{3}}{3}\tan^{-1}\left(\frac{x-1}{\sqrt{3}}\right) + C}$$