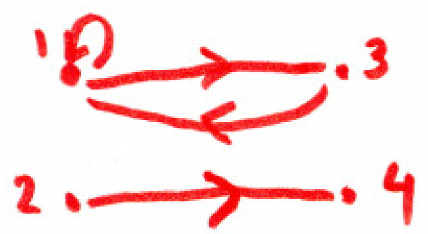
Relations and Digraphs

def A relation on a set A is a set $R \subseteq A \times A$ of ordered pairs of A.

Ex: $A = \{1, 2, 3, 4\}$

$R = \{(1,1), (1,3), (3,1), (2,4)\}$



def A directed graph, or digraph, D consists of a vertex set $V(D)$ and an edge set $E(D)$ that is a relation on the vertex set.

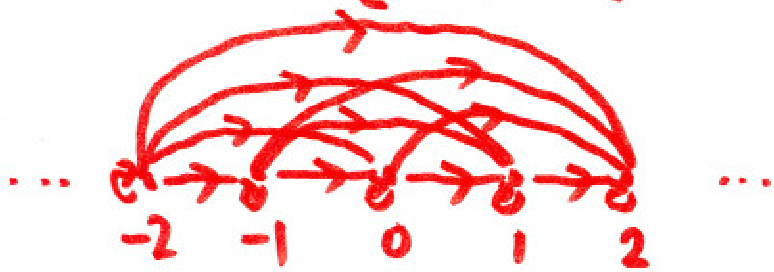
Note: Relations and digraphs give different languages for talking about the same objects.

Which language we prefer depends upon context.

Notation: If $R \subseteq A \times A$ is a relation on A, we write aRb for $(a,b) \in R$

More Examples:

- $A = \mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
- $R = \leq = \{(a,b) \mid a \leq b\}$



$-1 \leq 2$ or $-1R2$
or
$R(-1,2)$ or $\leq(-1,2)$

- $A = \{1, 2, \ldots, n\} = [n]$
- $R = \{(a,b) \mid a-b \text{ is even}\}$

- $A = \mathcal{P}([n]) = \{a \mid a \subseteq \{1,2,\ldots,n\}\}$
- $R = \{(a,b) \mid a \subseteq b\}$

def A relation R on A is

- **reflexive**, if $\forall a \in A \;\; aRa$
- **transitive**, if $\forall a,b,c \in A \;\; aRb$ and bRc imply aRc
- **symmetric**, if $\forall a,b \in A \;\; aRb \Longleftrightarrow bRa$
- **antisymmetric**, if $\forall a,b \in A \;\; aRb$ and bRa imply $a=b$

def A relation R on A is an **equivalence relation** if R is reflexive, transitive, and symmetric.

Note: We often use $\sim$ to denote an equivalence relation. Dynamic equivalence relations are implemented on computers with the Union-Find data structures and algs.

Examples of Equivalence Relations

Ex 1 · $A = \mathcal{P}([n])$,

· $\sim = \{(a,b) \mid |a| = |b|\}$ or

$a \sim b \iff |a| = |b|$

- **Reflexive:** $\forall a \ |a| = |a|$ ✓
- **Transitive:** $\forall a,b,c \ |a| = |b|$ and $|b| = |c|$ imply $|a| = |c|$ ✓
- **Symmetric:** $\forall a,b \ |a| = |b| \iff |b| = |a|$ ✓

If $n = 3$, then $A = \mathcal{P}([3]) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

and the picture is:

Ex 2: Let G be a graph.

- $A = V(G)$
- $u \sim v \iff \exists$ uv-walk in G

- Reflexive: $\forall u \in V(G)$, $W = u$ is a uu-walk in G ✓

- Transitive: $\forall u, v, w \in V(G)$ if G contains a uv-walk W_1 and a vw-walk W_2, then adding W_2 to the end of W_1 gives a uw-walk W in G:

$$W = \underbrace{u \cdots v}_{W_1} \cdots w$$

$$\underbrace{v \cdots w}_{W_2}$$

✓

- Symmetric: $\forall u, v \in V(G)$ if G contains a uv-walk W_1, then reversing W_1 gives a vu-walk in G. ✓

def If $\sim$ is an equivalence relation on A, then we define for each $a \in A$ the equivalence class of a, denoted $[a]$, by

$$[a] = \{ x \in A \mid a \sim x \}$$

Thm If $a \sim b$, then $[a] = [b]$.

Pf: $x \in [a] \iff a \sim x$. Now $a \sim b \Rightarrow b \sim a$ by the symmetric property of $\sim$. Transitivity and $b \sim a$, $a \sim x$ imply $b \sim x$ and hence $x \in [b]$. Therefore $[a] \subseteq [b]$. A similar argument shows $[b] \subseteq [a]$. Hence $[a] = [b]$. ∎

Cor Either $[a] \cap [b] = \emptyset$ or $[a] = [b]$.

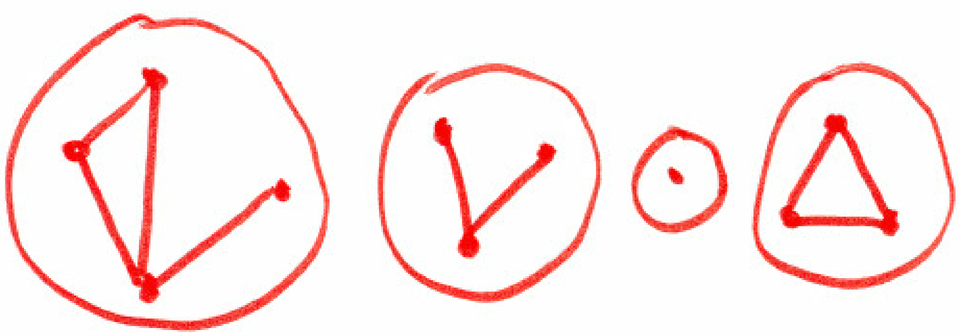
Pf: If $c \in [a] \cap [b]$, then $a \sim c$ and $b \sim c$. Hence $[a] = [c] = [b]$. ∎

Remark: The equivalence classes of $\sim$ form a partition of A: each $a \in A$ is in exactly one equivalence class.

def The **components** of a graph G are the equivalence classes of the relation

- $A = V(G)$
- $u \sim v \iff \exists$ uv-walk in G

Ex:



There are 4 components in the graph above; ~~each~~ they are circled.

def A graph G is **connected** if it has only one component.

Question: if G is a "large" connected graph, then G must contain some edges; how many must G have?

Strategy:

- Think about extreme cases:
  ⇒ If G has 0 edges, then G has many ($|V(G)|$) components
- The more components G has, the further away it is from being connected.
- Find a conjecture which interpolates between easy/extreme cases and the statement you want to prove
- Prove the conjecture by induction.
- Another example: sometimes it is easier to prove a stronger statement.

Thm Let G be an n-vertex graph with $0 \leq m \leq n-1$ edges. Then G has at least $n-m$ components.

Pf: By induction on m.

If $m=0$, then G has no edges and the components of G consist of n isolated vertices, so G has n components and the statement holds.

If $m \geq 1$, then G has an edge uv. Let H be the graph obtained from G by removing the edge uv; that is

- $V(H) = V(G)$
- $E(H) = E(G) - \{\{u,v\}\}$.

(In the future, we will write $H = G - uv$.)

Now H has $\overset{<m}{m-1}$ edges, so the inductive hypothesis implies that H has at least $n-(m-1) = n-m+1$ components.

If u and v are in the same component of H, then G has the same number of components as H and we are done.

If u and v are in different components of H, then adding the edge uv to H merges the components $[u]$ and $[v]$ to a single component in G. The other components remain unchanged. In this case, G has one fewer component than H, so G has at least $n-m$ components. ■

<u>Cor</u> If G is a connected n-vertex graph, then G has at least $n-1$ edges.

<u>Pf</u>: Suppose for a contradiction that G has at most $n-2$ edges. Then our theorem implies G has at least $n-(n-2)=2$ components, contradicting that G is connected. ■