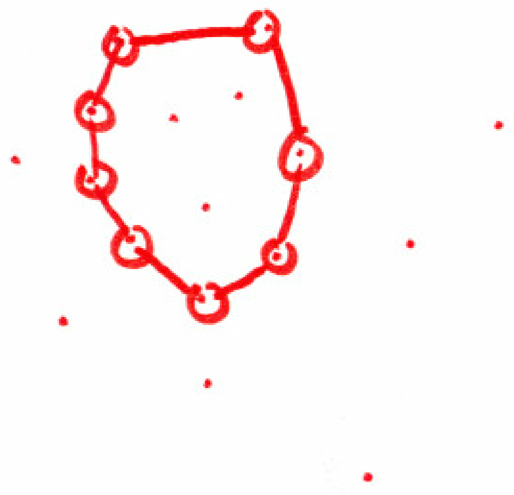
Ramsey Theory

- A very nice, clean, and surprising result
- A personal favorite
- In short, Ramsey Theory says:

"Large structures must contain highly ordered substructures."*

*(sometimes)

Ex:



If you draw many points in the plane, you can always find a set of points in convex position. The more points you draw, the larger the convex set you can be sure to find.

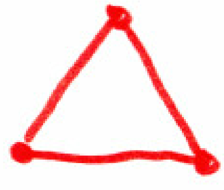
Ex: If enough people gather at your party, you'll be able to find a group of 10 people who are mutual friends or mutual strangers.

(Note: enough means somewhere in the range 798-23,556 according to Wikipedia.)

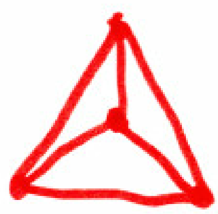
def The complete graph or clique, on n vertices, denoted K_n is the graph with all possible edges:

- $V(G) = [n] = \{1, 2, \ldots, n\}$
- $E(G) = \{\{u,v\} \mid u,v \in V(G)\}$

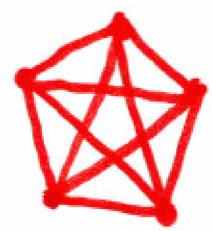
Ex:



K_3

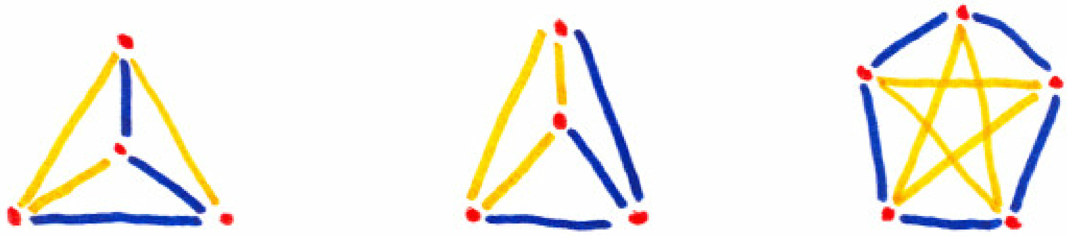


K_4



K_5

- Our Ramsey Theorem is about what must happen when we color the edges of K_r with two colors blue and yellow:



<u>Thm</u> For each $m, n \geq 1$ there exists r such that each way of coloring the edges of $G = K_r$ with <u>blue</u> and <u>yellow</u> leads to one of the following:

- A set $S \subseteq V(G)$ of m vertices, in which all edges are <u>blue</u>, or

- A set $S \subseteq V(G)$ of n vertices, in which all edges are <u>yellow</u>.

Note: This is a theorem whose statement contains many alternating quantifiers

$$\forall\, m,n \geq 1 \quad \exists\, r \quad \forall \text{ colorings} \quad \exists\, S \subseteq V(G)$$

$$\ldots\ldots$$

It can be confusing to keep track of what is going on.

Helpful Tip: Turn the theorem into a game, with two players:

- the prover, which moves at $\exists$ (there exists) steps, and
- the adversary, which moves at $\forall$ (for all) steps.

If the prover can force a win, the theorem is proved.

Ex: Our Ramsey Theorem as a game:

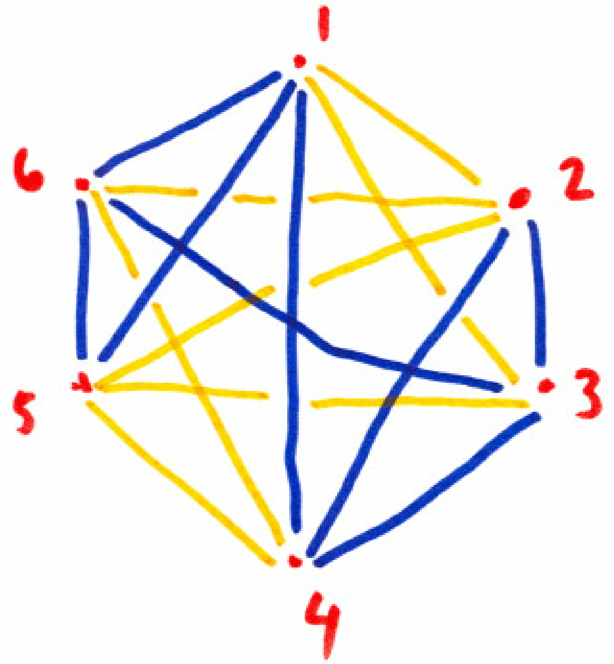
- We will be the prover.
- Because the first quantifier is $\forall$, our opponent goes first.

1. The adversary chooses $m, n \geq 1$.
   (Ex: "I pick $m=3$ and $n=3$" -- Adversary)

- The next quantifier is $\exists$, so it is our turn to move:

2. We choose r. We probably need to pick r differently, depending upon how the adversary selected m and n.
   (Ex: "Ok, then I will pick $r=6$" -- prover)

3. Our adversary chooses how to color the edges of K_6;
   (Ex: "Ok, how about this coloring?")



4. We choose an S with 3 vertices that has the desired property.
(Ex: "Aha! Pick $S = \{2, 3, 4\}$. Because all edges with endpoints in S are blue, we win." --prover)

<u>Remark:</u> We have seen that if the adversary picks $m = n = 3$, then we can respond with $r = 6$ and win the game:

"Any party with 6 people contains 3 <u>mutual friends</u> or 3 <u>mutual strangers</u>."

Thus, the theorem is true for the case $(m, n) = (3, 3)$.

Thm $\forall m,n \geq 1\ \exists r\ \forall$ blue/yellow-colorings of the edges of K_r, $\exists S \subseteq V(K_r)$ such that

- $|S| = m$ and all edges in S are blue, or
- $|S| = n$ and all edges in S are yellow.

Pf: By induction.

m \ n	1	2	3	4	5	$\cdots$	n
1	1	1	1	1	1		1
2	1	4	5	6	7		
3	1	5	6	7	8		
4	1	6	7	8	9		
5	1	7	8	9	10		
$\vdots$	$\vdots$					$\ddots$	
m	1						m+n

Our size function

- Problem set is $\{(m,n) \mid m,n \geq 1\}$ $= \{1,2,3,\ldots\} \times \{1,2,3,\ldots\}$ $= \{1,2,3,\ldots\}^2$

$$\text{size}((m,n)) = \begin{cases} 1 & m=1 \text{ or } n=1 \\ m+n & \text{otherwise} \end{cases}$$

Base Case: Our problems of minimum size are the ~~pairs~~ ordered pairs (m,n) where $m=1$ or $n=1$.

Suppose the adversary picks (m,n) and $m=1$ or $n=1$. We respond by choosing $r=1$. Note that no matter how the adversary colors the edges of $K_1 = \bullet$, we can choose S to be the vertex in K_1. Now, if $m=1$, then $|S|=1$ and all edges in S are blue, so we win. Otherwise, if $n=1$, then $|S|=1$ and all edges in S are yellow, so we win in this case also.

__Inductive Step__: Suppose the adversary picks (m, n) with $m \geq 2$ and $n \geq 2$.

Note that $(m-1, n)$ and $(m, n-1)$ are problem instances of __smaller__ __size__.

Therefore, by the __inductive__ __hypothesis__, there exists r_1 such that no matter how the adversary colors K_{r_1}, we can find S so that

- $|S| = m-1$ and all edges in S are __blue__, or
- $|S| = n$ and all edges in S are __yellow__.

Similarly, by the __inductive__ __hypothesis__, there exists r_2 so that no matter how the adversary colors K_{r_2}, we can find a __blue__ S of size m or a __yellow__ S of size $n-1$.

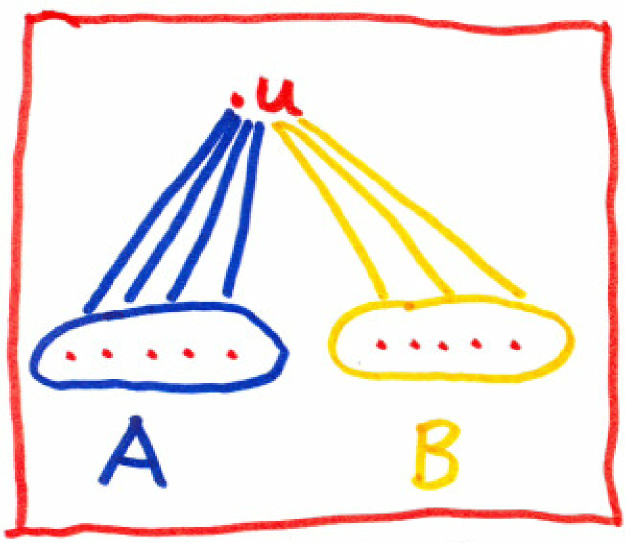
So far, our adversary has picked (m,n) and we have not responded.

We are now ready to answer; we choose $r = r_1 + r_2$.

Next, it is our adversary's turn to choose a coloring of K_r.

Now, it is our turn to find a blue S of size m or a yellow S of size n.

Let u be an arbitrary vertex, and define



$$A = \{v \mid uv \text{ is blue}\}$$

$$B = \{v \mid uv \text{ is yellow}\}$$

Note that either $|A| \geq r_1$ or $|B| \geq r_2$. (Otherwise, $r = |A| + |B| + 1$
$\leq r_1 - 1 + r_2 - 1 + 1$
$\leq r_1 + r_2 - 1$

contradicts $r = r_1 + r_2$.)

Suppose first that $|A| \geq r_1$. The vertices in A contain a copy of K_{r_1} which our adversary has colored. Therefore we can find $S_0 \subseteq A$ so that

- $|S_0| = m-1$ and all edges in S_0 are <u>blue</u>, or

- $|S_0| = n$ and all edges in S_0 are <u>yellow</u>

In the first case, we choose $S = S_0 \cup \{u\}$ and we win because S has size m and is <u>blue</u>. In the second case, we chose $S = S_0$ and we win because S has size n and is <u>yellow</u>.

Otherwise, if $|B| \geq r_2$, then we can find a <u>blue</u> $S_0 \subseteq B$ of size m (and we win with $S = S_0$), or we can find a <u>yellow</u> $S_0 \subseteq B$ of size $n-1$, (and we win with $S = S_0 \cup \{u\}$). $\blacksquare$

def For each $m, n \geq 1$ define $\underline{R(m,n)}$ to be the smallest integer r so that each blue/yellow coloring of the edges of K_r contains a <u>blue</u> $S \subseteq V(K_r)$ of size m or a <u>yellow</u> $S \subseteq V(K_r)$ of size n.

<u>Note</u>: Our proof can be modified to show that

$$R(m,n) \leq \binom{m+n}{m} = \binom{m+n}{n} \leq 2^{m+n}$$

On Exam 1, I will ask you to describe the necessary modifications.