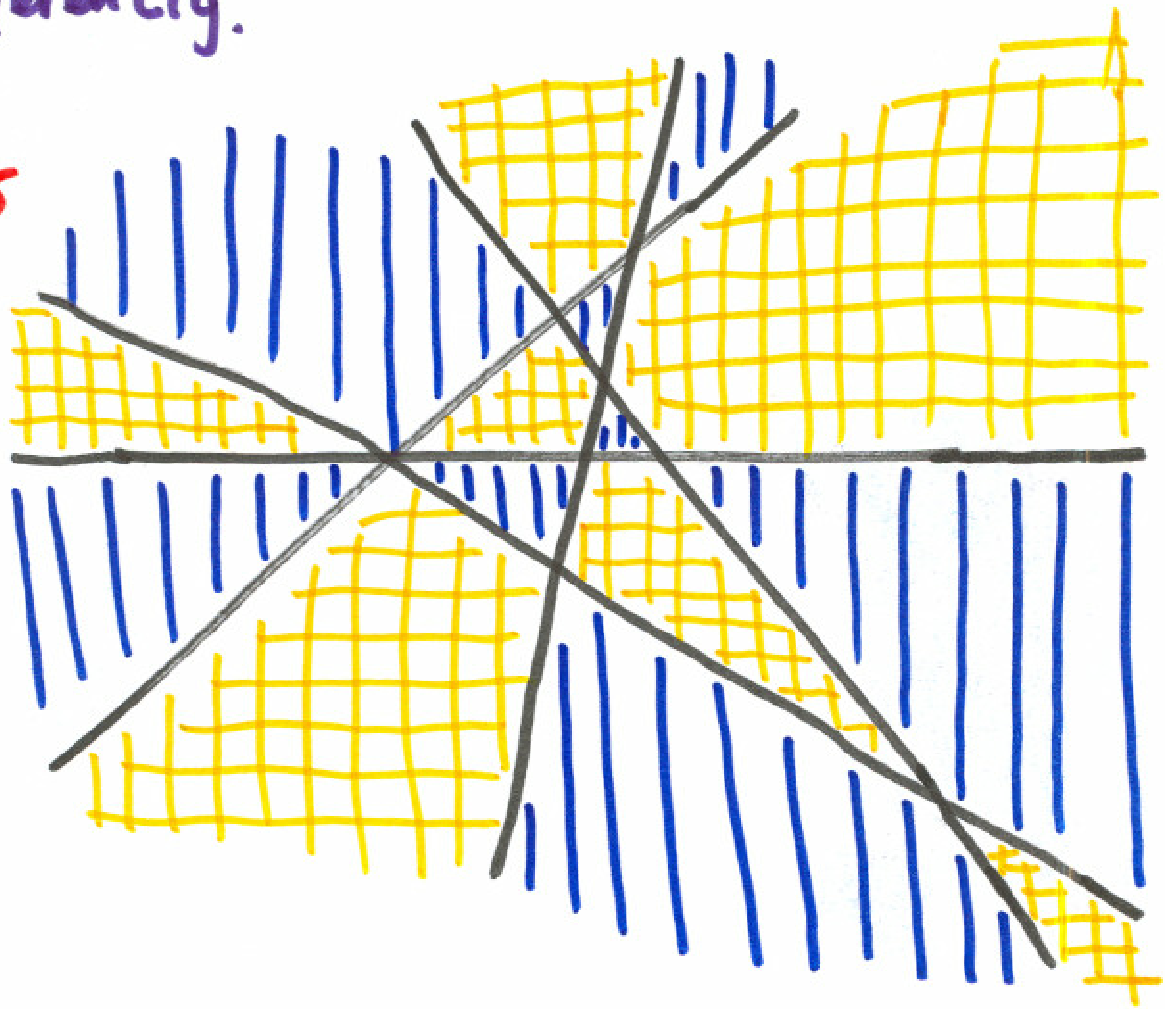
<u>Thm</u> Consider an arrangement of n lines in the plane. The regions produced by the lines can be colored blue and yellow so that neighboring regions are colored differently.

<u>Ex</u>: $n=5$



Pf: By induction on n. (Implicit def of size)

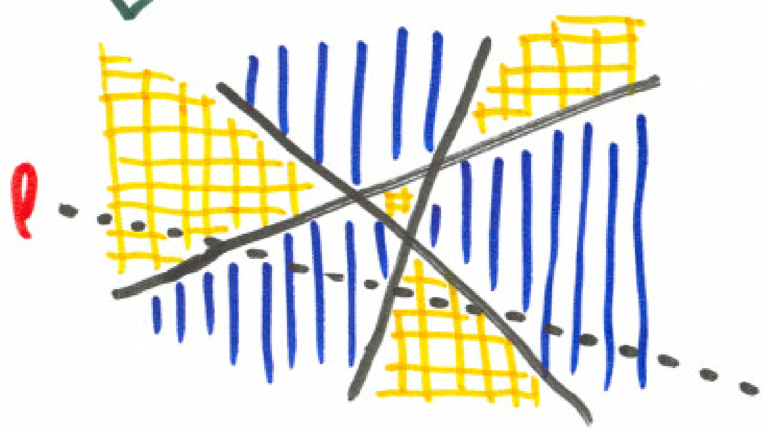
Base Case

If $n=0$, there is only one region (the entire plane), which we can color blue or yellow.

Inductive Step

Suppose $n \geq 1$, and let L be the set of n lines. Choose $\ell \in L$ arbitrarily and let $L' = L - \{\ell\}$.

Because $|L'| = n-1 < n$, obtain a blue/yellow coloring of the regions produced by L' by invoking the inductive hypothesis.



Coloring of L' via the inductive hypothesis

We use this coloring to produce a blue/yellow coloring of the regions produced by L as follows:

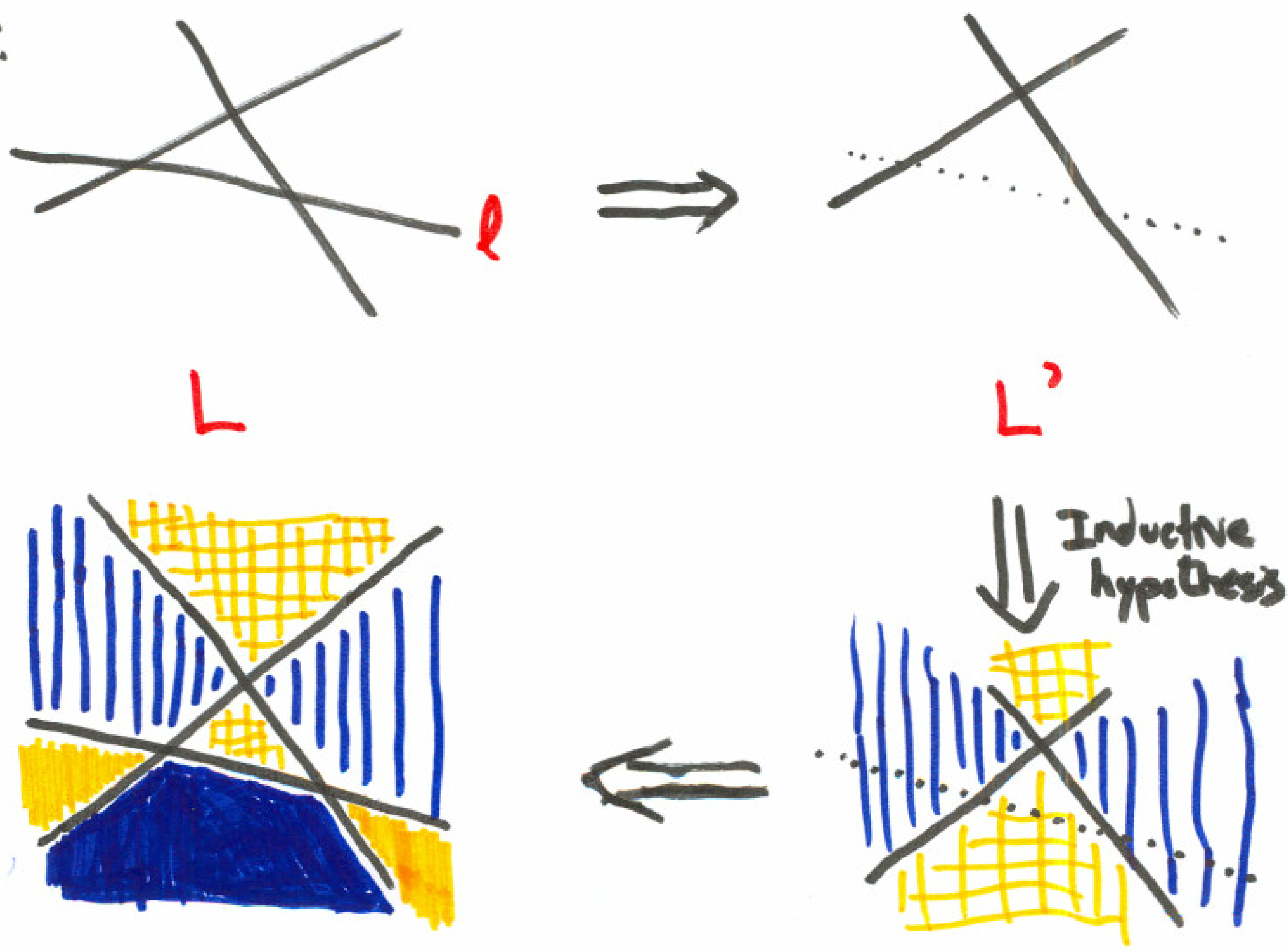
Flip the color of each region to one side of ℓ.

Neighboring regions that are on the same side of ℓ receive different colors, because the color on both regions either stayed the same or flipped.

Regions that are neighbors across ℓ receive different colors, by construction.

Therefore neighboring regions receive different colors. ∎

Ex:



Consider the following game. A sequence of coins is placed on a table. A <u>move</u> consists of removing a coin with heads face up* and flipping any neighboring coins that remain.

The game is won if all coins are removed.

<u>Ex</u>:

T	H	T	H	H	T
H	_	H	H	H	T
_	_	H	H	H	T
_	_	H	T	_	H
_	_	H	T	_	_
_	_	_	H	_	_
_	_	_	_	_	_

It is not possible to win starting from

T	H	H	H	H	T

Problem: from which sequences is it possible to win?

Strategy:

- Try many, many small examples
- Build intuition about the problem
- Conjecture about the solution
- Try to prove it

Thm A non-empty sequence $S = \underline{x_1}\ \underline{x_2} \cdots \underline{x_n}$ is winnable if and only if it contains an odd number of heads.

Pf: By induction on n.

If $n=1$, then $S = \underline{H}$ or $S = \underline{T}$; in either case, the theorem holds.

Suppose $n \geq 2$. We consider two cases, depending upon whether the number of heads in S is even or odd.

<u>Case 1</u>: The number of heads in S is odd.

In this case, S contains a head; suppose the first head occurs at x_k, ~~for~~ $1 \leq k \leq n$.

$$\begin{array}{cccccccccc} T & T & T & \cdots & T & H & ? & ? & \cdots & ? \\ x_1 & x_2 & x_3 & & x_{k-1} & x_k & x_{k+1} & x_{k+2} & & x_n \end{array}$$

Removing x_k and flipping its neighbors results in the ~~sequence~~ configuration

$$\begin{array}{cccccccccc} T & T & T & \cdots & H & \_ & ¿ & ? & \cdots & ? \\ x_1 & x_2 & x_3 & & x_{k-1} & x_k & \neg x_{k+1} & x_{k+2} & & x_n \end{array}.$$

By removing and flipping $x_{k-1}, x_{k-2}, \ldots, x_1$ in order, we can remove the first k coins, resulting in the configuration

$$\begin{array}{ccccccc} \_ & \_ & \_ & \cdots & \_ & \_ & \\ x_1 & x_2 & x_3 & & x_{k-1} & x_k & \end{array} \underbrace{\begin{array}{cccc} ¿ & ? & \cdots & ? \\ \neg x_{k+1} & x_{k+2} & & x_n \end{array}}_{S'}.$$

Let S' be the sequence of coins that remains. That is, S' is obtained from S by throwing away x_1 through x_k and flipping x_{k+1} (if it exists).

If S' is empty, then S is winnable as required.

If S' is non-empty, then either S' has the same number of heads as S (if $x_{k+1} = T$), or S' has two fewer heads than S (if $x_{k+1} = H$). Either way, S' has an odd number of heads.

Because S' is a sequence of $1 \leq n-k < n$ coins with an odd number of heads, the <u>inductive hypothesis</u> implies that S' is winnable, and we can remove the remaining coins. Therefore S is winnable.

Case 2: The number of heads is even.

Suppose for a contradiction that S is winnable. Recalling that S is non-empty, let x_k be the first heads removed in some winning sequence.

$$\underbrace{\underset{x_1}{?}\ \ \underset{x_2}{?}\ \ \cdots\ \ \underset{x_{k-1}}{?}}_{S_1}\ \ \underset{x_k}{H}\ \ \underbrace{\underset{x_{k+1}}{?}\ \ \cdots\ \ \underset{x_n}{?}}_{S_2}$$

call the left one S_1 and the right one S_2. Because S has an even number of heads, S_1 and S_2 have an odd number of heads in total. It follows that exactly one of S_1, S_2 has an odd number of heads.

If S_1 has an odd number of heads, then S_1 is not empty and flipping x_{k-1} results in a non-empty sequence of coins S' of size $1 \leq k-1 \leq n$ with an even number of heads. By the <u>inductive hypothesis</u>, S' is not winnable. But x_k is the first heads removed from S in a winning sequence, so S' must be winnable. This is a contradiction.

The case that S_2 has an odd number of heads is similar, and also leads to a contradiction.

Therefore S is not winnable and the theorem holds. ∎