Graphs

- **Graphs** are used to describe relationships between objects.

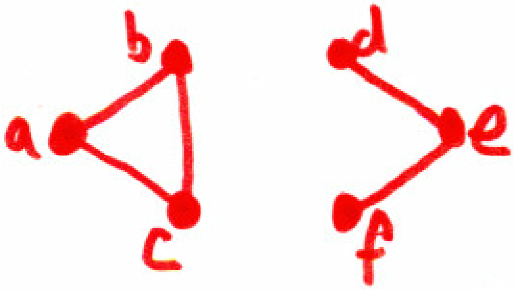
def A **graph** G is a pair of sets:

- A set $V(G)$ of <u>vertices</u>
- A set $E(G)$ of <u>edges</u>

such that each edge $e \in E(G)$ is a subset of $V(G)$ of size 2.

Note: Our graphs are finite unless we say otherwise.

Ex:
- $V(G) = \{a, b, c, d, e, f\}$
- $E(G) = \{\{a,b\}, \{b,c\}, \{a,c\}, \{d,e\}, \{e,f\}\}$



- $V(G) = \{ x \mid x \text{ is a city} \}$
- $E(G) = \{ \{x,y\} \mid \text{there are direct flights between } x \text{ and } y \}$

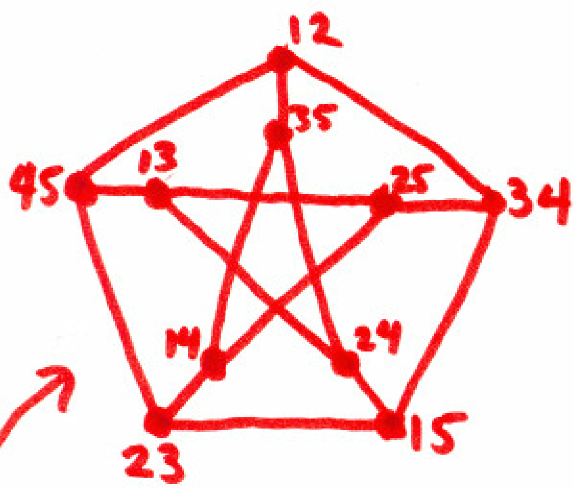
- $V(G) = \{ x \mid x \text{ is a soccer team} \}$

  $E(G) = \{ \{x,y\} \mid x \text{ and } y \text{ have played a game} \}$

<u>But Not</u>: $E(G) = \{ \{x,y\} \mid x \text{ has beaten } y \}$

<u>Q</u>: What is wrong?

<u>A</u>: "x has beaten y" is a property of the ordered pair (x,y), not the pair $\{x,y\}$.



- $V(G) = \{ A \subseteq [5] \mid |A| = 2 \}$
- $E(G) = \{ \{A,B\} \mid A \cap B = \emptyset \}$

This graph has a name: the <u>Petersen Graph</u>

Graph Terminology

- Let G be a graph.
- If $u, v \in V(G)$ and $\{u,v\} \in E(G)$, we say that u and v are adjacent or neighbors.
- If $e \in E(G)$ and $e = \{u,v\}$, we say that u and v are endpoints of e, and that e is incident to u.

Instead of writing $e = \{u,v\}$, we may simply write $e = uv$. (Compare with our notation for permutations.)

- For each vertex v in G, the neighborhood of v, denoted $N(v)$, is

$$N(v) = \{u \in V(G) \mid u \text{ and } v \text{ are adjacent}\}$$

- An isolated vertex is a vertex v with $N(v) = \emptyset$
- A dominating vertex is a vertex v with $N(v) = V(G) - \{v\}$

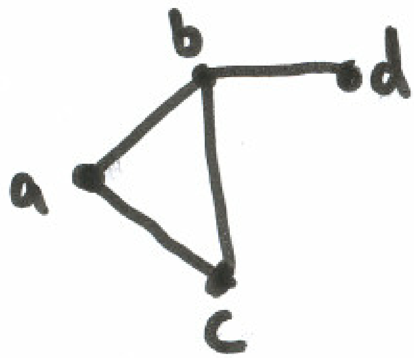
def If G is a graph and $v \in V(G)$, then the **degree** of v, denoted $\underline{d(v)}$, is $|\{e \in E(G) \mid v \in e\}|$.

Thm If G is a graph and $V(G) = \{v_1, \ldots, v_n\}$, then $d(v_1) + d(v_2) + \cdots + d(v_n) = 2|E(G)|$.

(Equivalently, $\sum_{j=1}^{n} d(v_j) = 2|E(G)|$ or

$$\sum_{v \in V(G)} d(v) = 2|E(G)|.)$$

Ex:



$d(a) + d(b) + d(c) + d(d)$
$=$
$2 + 3 + 2 + 1$
$=$
$8 = 2|E(G)|$

$V(G) = \{a, b, c, d\}$

$E(G) = \{\{a,b\}, \ldots\}$

Thm If G is a graph, then $\sum_{v \in V(G)} d(v) = 2|E(G)|$.

Pf: Consider $e \in E(G)$. We have that $e = \{u, v\}$ (or simply $e = uv$) for two vertices $u, v \in V(G)$. Hence, e contributes once to $d(u)$, once to $d(v)$, and zero to the degree of other vertices.

Therefore each edge contributes 2 to the sum $\sum_{v \in V(G)} d(v)$. ∎

The Pigeonhole Principle

If more than n objects are placed into n bins, some bin must contain more than one object.

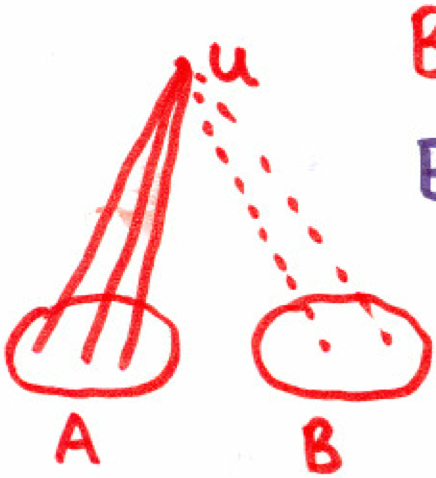
Compare: if $f : A \to \{1, 2, \ldots, n\}$ is injective, then $|A| \le n$.

A Classic Application

In any party with at least 6 people, you can find a group of 3 mutual friends or a group of 3 mutual strangers.

Thm If G is a graph on 6 vertices, there is a set $S \subseteq V(G)$ with $|S| = 3$ such that either the vertices in S are pairwise adjacent, or the vertices in S are pairwise non-adjacent, (i.e. S is an independant set).

Pf: Let $u \in V(G)$, $A = \{v \in V(G) \mid v \in N(u)\}$, $B = \{v \in V(G) \mid v \neq u$ and $v \notin N(u)\}$.

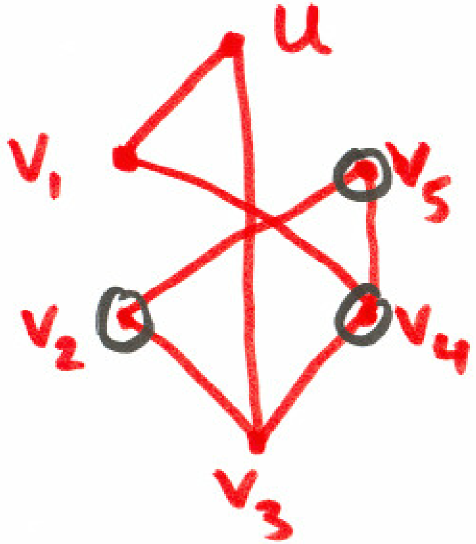


Either $|A| \geq 3$ or $|B| \geq 3$. In the first case, if A contains an adjacent pair of vertices $\{v, w\}$ then $S = \{u, v, w\}$ is pairwise adjacent.

Otherwise, $S = A$ is an independent set.

In the second case, if $|B| \geq 3$, if B contains a pair $\{v, w\}$ of non-adjacent vertices, then $S = \{u, v, w\}$ is an independent set. Otherwise, $S = B$ is pairwise adjacent. ∎

~~Practice~~ Ex:



- $A = \{v_1, v_3\}$
- $B = \{\underline{v_2}, \underline{v_4}, \underline{v_5}\}$
- $\{v_2, v_4\} \subseteq B$ is a non-adj. pair

$\Longrightarrow S = \{u, v_2, v_4\}$ is an independent set.

Proof by Contradiction

- Suppose you want to prove a statement P.
- If, under the assumption "P is false", you can ~~prove~~ find ~~another~~ a statement Q and prove both
    - "Q is true"
    - "Q is false"

  then P must be true.

Lemma: If G is an n-vertex graph with $n \geq 2$, then either G has no isolated vertices or G has no dominating vertices. (or both).

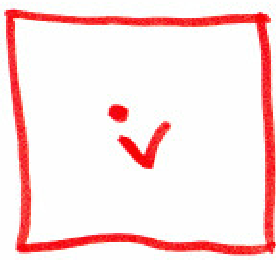
Pf: Suppose for a contradiction, that G contains an isolated vertex u and a dominating vertex v. Because u is isolated, u and v are not adjacent. Because v is dominating, u and v are adjacent. This is a contradiction. ∎

Question: Which statement in the proof uses $n \geq 2$?

- It must be used somewhere, or else the proof would establish

  P = "Every graph G either does not contain isolated vertices or does not contain dominating vertices."

- Note: P is false:



G

$V(G) = \{v\}$

$E(G) = \emptyset$

G is a graph with an isolated vertex and a dominating vertex.

- What goes wrong?

- Ans: the statement "Because v is dominating, u and v are adjacent" requires $u \neq v$.

  The proof uses $n \geq 2$ in the implicit assumption $u \neq v$.

- Better: Explicitly say "Because $n \geq 2$, $u \neq v$."

Thm Every graph G with at least two vertices contains two vertices of the same degree.

Pf: Let $B = \{d(v) \mid v \in V(G)\}$, and let $n = |V(G)|$. Because $n \geq 2$, our lemma implies that either

(1) $0 \notin B$, or

(2) $n-1 \notin B$.

In the first case, $B \subseteq \{1, 2, \ldots, n-1\}$. In the second case, $B \subseteq \{0, 1, \ldots, n-2\}$. In either case, B is a subset of a set of size $n-1$.

Hence $|B| \leq n-1$. View $V(G)$ as a set of objects and B as a set of bins; place ~~a vertex into the~~ vertices into bins according to degree.

Because $|V(G)| = n$ and $|B| \leq n-1$, the <u>pigeonhole principle</u> implies the result. ∎

More Graph Terminology

- Let G be a graph.

- A **walk** in G is a list

$$W = w_1 w_2 w_3 \cdots w_k$$

of vertices such that $\forall\, 1 \le j \le k-1$ $w_j w_{j+1} \in E(G)$. The **length** of W is $k-1$, w_1 and w_k are the **endpoints** of W, and we say that W is a **$w_1 w_k$-walk**. We say W is **closed** if $w_1 = w_k$.

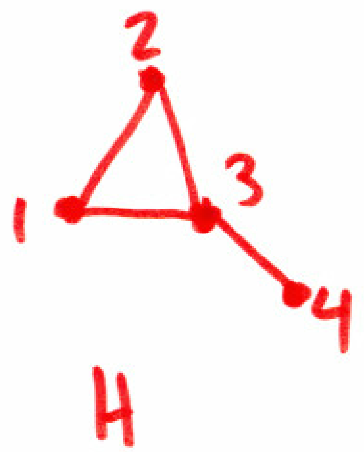
- A **trail** in G is a walk $W = w_1 w_2 \cdots w_k$ with the property that all edges are distinct:

$$\forall\, 1 \le i < j \le k-1 \quad \{w_i, w_{i+1}\} \ne \{w_j, w_{j+1}\}$$

- A **path** in G is a walk $P = w_1 w_2 \cdots w_k$ with the property that all vertizes are distinct:

$$\forall\, 1 \le i < j \le k \quad w_i \ne w_j$$

Ex



- 1 3 4 2 3 is not a walk
- 1 3 4 3 2 is a walk, but not a trail
- 4 3 2 1 3 is a trail, but not a path
- 4 3 1 2 is a path