#11

|| Let ~~X~~ X be the # of bins with ≥ 1 || ball.

- If $m=1$ then always $X=1$, so $E[X]=1$
- ✓ If $m=0$, " " $X=0$, so $E[X]=0$
- If $m\geq 1$ and $n=1$, $X=1$, so $E[X]=1$

For each $1\leq j\leq n$ let $X_j = \begin{cases} 1 & j\text{th bin has a ball} \\ 0 & \text{otherwise} \end{cases}$

Because $X = X_1 + X_2 + \cdots + X_n$, so linearity of expectation gives

$$E[X] = E[X_1] + E[X_2] + \cdots + E[X_n].$$

What is $E[X_j]$? By defn,

$$E[X_j] = 0 \cdot Pr(X_j = 0) + 1 \cdot Pr(X_j = 1)$$
$$= Pr(X_j = 1)$$

Note $X_j = 1 \iff$ the jth bin contains ≥ 1 ball.

Using the complimentary event $X_j = 0$, we have

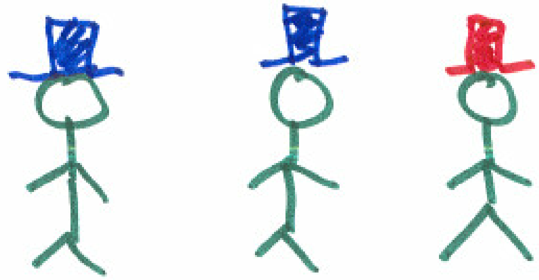
$$\begin{aligned}
Pr(X_j = 1) &= 1 - Pr(X_j = 0) \\
&= 1 - Pr(\text{1st ball misses bin } j \cap \\
&\qquad\quad \text{2nd ball misses bin } j \cap \\
&\qquad\quad \vdots \\
&\qquad\quad \text{mth ball misses bin } j) \\
&= 1 - Pr(\text{1st ball misses}) \cdot \ldots \cdot Pr(\text{mth ball misses}) \\
&= 1 - \left(\tfrac{n-1}{n}\right) \cdot \ldots \cdot \left(\tfrac{n-1}{n}\right) \\
&= 1 - \left(\tfrac{n-1}{n}\right)^m
\end{aligned}$$

Therefore $\boxed{E[X] = n\left(1 - \left(\frac{n-1}{n}\right)^m\right)}$

- Using $1 - x \approx e^{-x}$ for x close to 0,

$$E[X] = n\left(1 - \left(1 - \tfrac{1}{n}\right)^m\right) \approx n\left(1 - \left(e^{-\frac{1}{n}}\right)^m\right) = n\left(1 - e^{-\frac{m}{n}}\right)$$

#2



A B C

no guess   no guess   guess

$\Rightarrow$ Group wins

Using this strategy, the group wins $\Longleftrightarrow$ both colors are used

$\Longleftrightarrow$ not

or

So $Pr(win) = \frac{6}{8} = \frac{3}{4}$

Strategy:

- If a person sees hats of different colors, then $\boxed{\text{no guess}}$
- If a person sees , guess
- If a person sees , guess

#3 Let X be the # of interviews scheduled.

For $1 \le j \le n$, let $X_j = \begin{cases} 1 & j\text{th (best) candidate interviewed} \\ 0 & \text{otherwise} \end{cases}$

Because $X = X_1 + \cdots + X_n$,

$$E[X] = E[X_1] + E[X_2] + \cdots + E[X_n]$$

$$= Pr(X_1 = 1) + Pr(X_2 = 1) + \cdots + Pr(X_n = 1)$$

What is $Pr(X_j = 1)$? I.E. in a uniformly random permutation π of $\{1, 2, \ldots, n\}$, what is the probability that j appears before $\{1, 2, \ldots, j-1\}$ in the permutation.

Key observation: throwing away $\{j+1, j+2, \ldots, n\}$ from π yields a uniformly random permutation of $\{1, \ldots, j\}$.

Hence $Pr(X_j = 1) = 1/j$, and $E[X] = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n} = H_n \approx \log n$.

5

#5) Suppose we assign pirates to dinner time slots ~~with~~ via the following random process.

First choose a perm. of the pirates uniformly at random.

Split the pirates into 3 groups ~~at~~ depending upon which ~~group one is~~ third the pirate is in: first third, middle third, or last third.

Let X be the number of bad pairs which eat together, and let $X_j = \begin{cases} 1 & \text{if } j\text{th bad pair eats together} \\ 0 & \text{otherwise} \end{cases}$

Then $E[X] = \sum_{j=1}^{k} E[X_j] = \sum_{j=1}^{k} Pr(X_j = 1)$

$$= \sum_{j=1}^{k} \frac{n/3 - 1}{n-1} = k\left(\frac{n/3 - 1}{n-1}\right).$$

Let A be the event that the stronger pirate in the jth pair eats in the first timeslot

B " " " " " eats in the second time slot

C " eats in the third time slot

$$Pr(X_j=1) = Pr(X_j=1 \mid A)\cdot Pr(A) + Pr(X_j=1 \mid B)\cdot Pr(B) + Pr(X_j=1 \mid C)\cdot Pr(C)$$

$$= \frac{n/3-1}{n-1}\cdot\frac{1}{3} + \frac{n/3-1}{n-1}\cdot\frac{1}{3} \cdots$$

$$= \frac{n/3-1}{n-1}$$