Random Variables

· A Random Variable describes a numerical property of the outcomes in our sample space.

def Let Ω be a sample space ~~of~~ a probability space. A random variable is a function $X : \Omega \to \mathbb{R}$ from Ω to the real numbers.

Ex "Suppose we flip a fair coin 3 times. Let X be the number of times that H (heads) is flipped."

Ω = { ·HHH ·THH ·HHT ·THT ·HTH ·TTH ·HTT ·TTT }

ω	$X(\omega)$
HHH	3
HHT	2
HTH	2
HTT	1

ω	$X(\omega)$
THH	2
THT	1
TTH	1
TTT	0

<u>Remark</u>: If X is a r.v. (random variable) we often use "$X = a$" for the event $\{\omega \in \Omega \mid X(\omega) = a\}$ consisting of all outcomes on which X takes the value a.

· One of the most important properties of a r.v. is the <u>average</u> <u>value</u> or expectation.

<u>def</u> If $X : \Omega \to \mathbb{R}$ is a random variable, then the <u>expectation of X</u>, denoted $E[X]$ is given by

$$E[X] = \sum_a a \cdot P_r(X = a) .$$

<u>Remark</u>: To treat ~~[illegible]~~ more complex cases, people

who study probability define $E[X]$ as an integral. For our purposes of studying <u>discrete</u> <u>probability</u>, this definition suffices.

<u>Ex</u>: Continuing with our coin example,

$$E[X] = \sum_a a \cdot \Pr(X=a)$$

$$= 0 \cdot \Pr(X=0) + 1 \cdot \Pr(X=1) + 2 \cdot \Pr(X=2) + 3 \cdot \Pr(X=3)$$

$$= 0 \cdot \frac{1}{8} + 1 \cdot \frac{3}{8} + 2 \cdot \frac{3}{8} + 3 \cdot \frac{1}{8}$$

$$= \frac{12}{8}$$

$$= \boxed{\frac{3}{2}}$$

So the expected number of heads is $\frac{3}{2} = 1.5$.

Notice that no outcome ω actually gives us $X(\omega) = 1.5$ heads.

Linearity of Expectation

Prop Let $X : \Omega \to \mathbb{R}$ ~~be~~ and $Y : \Omega \to \mathbb{R}$ be random variables. Define a new random variable $Z = X + Y$ by

$$Z(\omega) = X(\omega) + Y(\omega).$$

Then $E[Z] = E[X+Y] = E[X] + E[Y]$.

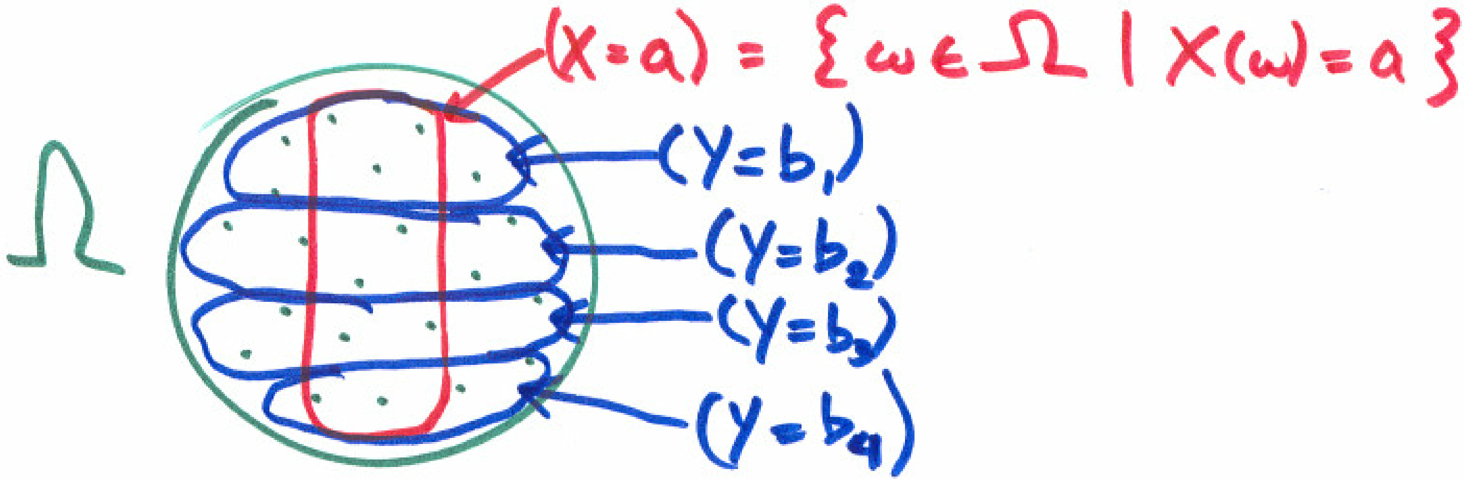
Proof: $\cancel{E[X+Y] =}$

$$E[X] + E[Y] = \sum_a a \cdot \Pr(X=a) + \sum_b b \cdot \Pr(Y=b)$$

$$= \sum_a a \cdot \left(\sum_b \Pr(X=a \cap Y=b) \right) +$$

$$\sum_b b \cdot \left(\sum_a \Pr(X=a \cap Y=b) \right)$$

$$= \sum_{a,b} a \cdot \Pr(X=a \cap Y=b) + \sum_{a,b} b \cdot \Pr(X=a \cap Y=b)$$


$(X=a) = \{\omega \in \Omega \mid X(\omega) = a\}$
$(Y=b_1)$
$(Y=b_2)$
$(Y=b_3)$
$(Y=b_4)$
Ω

$$= \sum_{a,b} (a+b) \cdot Pr(X=a \cap Y=b)$$

Now collecting all pairs (a,b) such that $a+b=c$ yields

$$= \sum_{c} c \cdot \cancel{Pr}\Big(\sum_{\substack{a,b \\ a+b=c}} Pr(X=a \cap Y=b) \Big)$$

$$= \sum_{c} c \cdot Pr(X+Y=c)$$

$$= \sum_{c} c \cdot Pr(Z=c)$$

$$= E[Z]$$

$\blacksquare$

<u>Cor</u> $E[X_1 + X_2 + \cdots + X_n] = E[X_1] + E[X_2] + \cdots + E[X_n]$

Linearity of Expectation can transform impossible computations into easy computations.

(Or at least <u>difficult</u> computations into <u>easier</u> computations.)

Ex: Derangements.

A total of ~~A party of~~ n people go to a concert and check their coats in the cloakroom. When they go to retrieve their coats, each person is handed one of the remaining coats at random.

Q: What is the expected number of people that receive their own coats back?

Model this problem with permutations:

$$\cdot\ \Omega = \{\pi \mid \pi \text{ is a perm. of } n \text{ elts}\}$$

where the outcome $\pi \in \Omega$ represents that $\forall j$ the jth person receives $\pi(j)$'s coat. Let X be the number of people that get their own coats back; i.e.

$$X(\pi) = |\{ j \mid \pi(j) = j \}|.$$

Ex: $n = 3$:

$\cdot 123$	$\cdot 231$
$\cdot 132$	$\cdot 312$
$\cdot 213$	$\cdot 321$

Ω

π	$X(\pi)$
123	3
132	1
213	1
231	0
312	0
321	1

So $E[X] = \sum_a a \cdot \Pr(X = a)$

$$= 0 \cdot \frac{2}{6} + 1 \cdot \frac{3}{6} + 3 \cdot \frac{1}{6}$$

$$= \frac{6}{6} = \boxed{1}.$$

For general n, ~~we~~ letting

$$A_a = \{ \pi \mid |\{ j \mid \pi(j) = j \}| = a \}$$
$$= \text{"}(X = a)\text{"}$$

so that A_a is the event that exactly a people get their own coat back, we have by definition

$$E[X] = \sum_{a=0}^{n} a \cdot \Pr(X=a)$$

$$= \sum_{a=0}^{n} a \cdot \Pr(A_a)$$

$$= \sum_{a=0}^{n} a \cdot \frac{|A_a|}{|\Omega|}$$

$$= \sum_{a=0}^{n} a \cdot \frac{|A_a|}{n!}$$

$$= \frac{1}{n!} \sum_{a=0}^{n} a \cdot |A_a|$$

$$= ????$$

- Linearity of Expectation provides an easier way.

- For $1 \le j \le n$ let X_j be the r.v. given by

$$X_j(\pi) = \begin{cases} 1 & \pi(j) = j \\ 0 & \text{otherwise} \end{cases}$$

These are called indicator random variables. Note that for each $\pi \in \Omega$,

$$X(\pi) = |\{j \mid \pi(j) = j\}|$$

$$= X_1(\pi) + X_2(\pi) + \cdots + X_n(\pi)$$

and so $X = X_1 + X_2 + \cdots + X_n$.

Therefore, by linearity of expectation,

$$E[X] = E[X_1] + E[X_2] + \cdots + E[X_n]$$

What is $E[X_j]$? By defn,

$$E[X_j] = 0 \cdot Pr(X_j = 0) + 1 \cdot Pr(X_j = 1)$$

$$= \Pr(X_j = 1)$$

$$= \Pr(\pi(j) = j)$$

$$= \frac{|\{\pi \mid \pi(j) = j\}|}{|\Omega|}$$

$$= \frac{(n-1)!}{n!}$$

$$= \frac{1}{n}.$$

Hence, $E[X] = E[X_1] + \cdots + E[X_n]$

$$= \frac{1}{n} + \cdots + \frac{1}{n}$$

$$= \boxed{1}$$

So the expected number of people that receive their own coats is 1 for all n, not just $n=3$.