Permutations

def A **permutation** of a set $\mathcal{U}$ is a bijection from $\mathcal{U}$ to $\mathcal{U}$.

Tips: After reading a definition, try to get an intuitive understanding.

Think: if $\pi : \mathcal{U} \to \mathcal{U}$ is a permutation of $\mathcal{U}$, then π is a bijection, so it sends each elt $j \in \mathcal{U}$ to a distinct element $\pi(j) \in \mathcal{U}$. Also, π "hits" each element in $\mathcal{U}$. Thus, π shuffles or permutes the elements of $\mathcal{U}$.

Ex: $\mathcal{U} = \{1, 2, 3\}$. There are 6 permutations of $\mathcal{U}$:

j	1	2	3
$\pi(j)$	1	2	3

j	1	2	3
$\pi(j)$	1	3	2

j	1	2	3
$\pi(j)$	2	1	3

j	1	2	3
$\pi(j)$	2	3	1

j	1	2	3
$\pi(j)$	3	1	2

j	1	2	3
$\pi(j)$	3	2	1

Note: if $\mathcal{U} = \{1, 2, \ldots, n\}$ we often express a permutation π of $\mathcal{U}$ by listing the values of π in order: $\pi(1), \pi(2), \ldots, \pi(n)$.

Ex: $\pi = 213$ instead of

j	1	2	3
$\pi(j)$	2	1	3

.

Question: if $\mathcal{U} = \{1, 2, \ldots, n\}$, how many permutations of $\mathcal{U}$ are there?

Intuition: to construct a permutation $\pi : \mathcal{U} \to \mathcal{U}$, first write down n blank spaces.

$$\pi = \_\_\ \_\_\ \_\_\ \_\_\ \_\_\ \_\_\ \cdots\ \_\_$$

Next, choose one of the n blank spaces for the 1.

$$\pi = \_\_\ \_\_\ \_\_\ \underline{1}\ \_\_\ \_\_\ \cdots\ \_\_$$

Next, choose one of the $n-1$ blank spaces for the 2.

$$\pi = \_\_\ \_\_\ \_\_\ \underline{1}\ \_\_\ \_\_\ \cdots\ \underline{2}$$

Repeating this process, we have $n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 1$

total choices available to us.

(1) Do different choices result in different permutations? YES.

(2) Does every permutation arise from choices we can make? YES.

Hence, our function from the set of total choices to the set of permutations of $\mathcal{U}$ is a bijection:

- (1) says it is injective
- (2) says it is surjective

After enough practice, this informal argument might be enough to convince you there are $n! = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 1$ permutations of $\mathcal{U}$.

How do we make it formal?
We need a set to "encode" our choices.

def $[n] = \{1, 2, \ldots, n\}$

Thm Let $n \geq 1$ be an integer, let $\mathcal{U} = [n]$, and define

$$A = [n] \times [n-1] \times [n-2] \times \cdots \times [1]$$

$$B = \{\pi \mid \pi \text{ is a permutation of } \mathcal{U}\}.$$

There is a bijection $f: A \to B$.

Pf: HW.

Corollary: $|B| = |A| = n!$

Binomial Coefficients

Question: How many ways can we select k elements from a set of size n?

def Let $n, k \geq 0$ be non-negative integers, let $\mathcal{U} = [n]$, and let
$\mathcal{A} = \{A \subseteq \mathcal{U} \mid |A| = k\}$.

We define $\binom{n}{k}$ (pronounced "n choose k") via $\binom{n}{k} = |\mathcal{A}|$.

Ex

- $\forall n \quad \binom{n}{0} = 1 \quad (\mathcal{A} = \{\emptyset\})$
- $k > n \Rightarrow \binom{n}{k} = 0 \quad (\mathcal{A} = \emptyset)$
- $\forall n \quad \binom{n}{1} = n \quad (\mathcal{A} = \{\{1\}, \{2\}, \dots, \{n\}\})$
- $\binom{4}{2} = 6 \quad (\mathcal{A} = \{\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}\})$

Rem: In HW1, we found an injection $f : \mathcal{A} \to \mathcal{U}^k$.
Hence, $\binom{n}{k} \leq n^k$.

Thm If $0 \le k \le n$, then $\binom{n}{k} = \binom{n}{n-k}$.

Pf: Let $\mathcal{U} = [n]$, and define

$$\mathcal{A} = \{ A \subseteq \mathcal{U} \mid |A| = k \}$$

$$\mathcal{B} = \{ B \subseteq \mathcal{U} \mid |B| = n-k \}.$$

By definition, $|\mathcal{A}| = \binom{n}{k}$ and $|\mathcal{B}| = \binom{n}{n-k}$.

Hence, it suffices to construct a bijection $f : \mathcal{A} \to \mathcal{B}$.

It is straightforward to check that

$$f(A) = \overline{A}$$

is the desired bijection. $\blacksquare$

Practice: For $n=5$ and $k=2$, write down $\mathcal{A}$, $\mathcal{B}$, and f.

Hint: $\binom{5}{2} = 10 = \binom{5}{3}$

Thm If $0 \le k \le n$, then $\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$.

Pf: Let $\mathcal{U} = [n+1]$ and $\mathcal{A} = \{A \subseteq \mathcal{U} \mid |A| = k+1\}$, so that $\binom{n+1}{k+1} = |\mathcal{A}|$.

Let $\mathcal{B} = \{A \in \mathcal{A} \mid n+1 \in A\}$ and let $\mathcal{C} = \{A \in \mathcal{A} \mid n+1 \notin A\}$.

Clearly, $\mathcal{B} \cup \mathcal{C} = \mathcal{A}$ and $\mathcal{B} \cap \mathcal{C} = \emptyset$. Therefore $|\mathcal{A}| = |\mathcal{B}| + |\mathcal{C}|$.

There are straightforward bijections

$$f : \mathcal{B} \to \{A \subseteq [n] \mid |A| = k\}$$
$$g : \mathcal{C} \to \{A \subseteq [n] \mid |A| = k+1\}$$

namely, $f(A) = A - \{n+1\}$ and $g(A) = A$.

Hence $|\mathcal{B}| = \binom{n}{k}$ and $|\mathcal{C}| = \binom{n}{k+1}$. The result follows. $\blacksquare$

Remark: The last theorem $\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$ tells us that the binomial coefficients $\binom{n}{k}$ is the kth entry of the nth row of Pascal's Triangle (indexing starts at zero):

$$
\begin{array}{l|ccccccccccc}
n=0 & & & & & & 1 & & & & & \\
n=1 & & & & & 1 & & 1 & & & & \\
n=2 & & & & 1 & & 2 & & 1 & & & \\
n=3 & & & 1 & & 3 & & 3 & & 1 & & \\
n=4 & & 1 & & 4 & & \boxed{6} & & \boxed{4} & & 1 & \\
n=5 & 1 & & 5 & & 10 & & \boxed{10} & & 5 & & 1 \\
 & \vdots & & & & & & & & & & \vdots
\end{array}
$$

$$6 + 4 = \binom{4}{2} + \binom{4}{3} = \binom{5}{3} = 10$$

<u>Notation</u>: Suppose $x_1, x_2, \ldots, x_n$ are numbers. It is cumbersome to write

$$x_1 + x_2 + \cdots + x_n, \quad x_1 \cdot x_2 \cdot \cdots \cdot x_n$$

for the sum and product, respectively.

Instead, we write

$$\sum_{j=1}^{n} x_j \quad \text{for} \quad x_1 + x_2 + \cdots + x_n$$

and

$$\prod_{j=1}^{n} x_j \quad \text{for} \quad x_1 \cdot x_2 \cdot \cdots \cdot x_n .$$

<u>Note</u>: the values of $\sum_{j=1}^{n} x_j$ and $\prod_{j=1}^{n} x_j$ depend only on $x_1, x_2, \ldots, x_n$, ~~and~~ <u>not</u> j.

<u>Think</u>: $\sum_{j=1}^{n} x_j$ and $\prod_{j=1}^{n} x_j$ are for loops.

If $A \subseteq \mathcal{U}$, we may use A as an <u>index set</u> and write

$$\sum_{j \in A} x_j \quad \text{for} \quad x_{j_1} + x_{j_2} + \cdots + x_{j_r}$$

where $A = \{j_1, j_2, \ldots, j_r\}$.

For example, if $A = \{2, 4, 6, \ldots, 2n\}$ ~~then~~ and $x_1, x_2, \ldots, x_{2n}$ are $2n$ numbers, then

$$\sum_{j \in A} x_j = x_2 + x_4 + x_6 + \cdots + x_{2n}$$

$$\prod_{j \in A} x_j = x_2 \cdot x_4 \cdot x_6 \cdot \cdots \cdot x_{2n}$$

Thm $\sum_{j=1}^{n} j = \binom{n+1}{2}$

(<u>Note:</u> $\sum_{j=1}^{n} j = 1+2+3+\dots+n$.)

<u>Pf</u>: Let $\mathcal{U} = [n+1]$, $\mathcal{A} = \{A \subseteq \mathcal{U} \mid |A| = 2\}$,

and for each $1 \le j \le n$, define

$$\mathcal{A}_j = \left\{ A \in \mathcal{A} \;\middle|\; \begin{array}{l} \text{the larger of the} \\ \text{two numbers in } A \\ \text{is } j+1 \end{array} \right\}$$

Note that if $i \neq j$ then $\mathcal{A}_i \cap \mathcal{A}_j = \emptyset$

and so $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n$ are pairwise disjoint.

Also, $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \dots \cup \mathcal{A}_n$ (or $\mathcal{A} = \bigcup_{j=1}^{n} \mathcal{A}_j$).

Therefore $|\mathcal{A}| = \sum_{j=1}^{n} |\mathcal{A}_j|$

$$\underbrace{|\mathcal{A}|}_{\binom{n+1}{2}} = \sum_{j=1}^{n} j$$

■

What is $\binom{n}{k}$ anyway?

- Let $0 \le k \le n$ be integers, $\mathcal{U} = [n]$, and $\mathcal{A} = \{A \subseteq \mathcal{U} \mid |A| = k\}$, so by definition, $\binom{n}{k} = |\mathcal{A}|$.

- To count $|\mathcal{A}|$, we will find a connection between $|\mathcal{A}|$ and permutations.

- Let $R = \{\pi \mid \pi \text{ is a permutation of } [n]\}$
  $S = \{\sigma \mid \sigma \text{ is a permutation of } [k]\}$
  $T = \{\tau \mid \tau \text{ is a perm. of } [n-k]\}$

<u>Thm</u> There is a bijection $f: \mathcal{A} \times S \times T \to R$.

(!!) <u>Cor</u>: $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$

<u>Pf</u>: $\binom{n}{k} k!(n-k)! = |\mathcal{A} \times S \times T| = |R| = n!$ ∎

Thm There is a bijection $f : \mathcal{A} \times S \times T \to R$.

Pf: Consider an elt $(A, \sigma, \tau) \in \mathcal{A} \times S \times T$. We must choose a permutation $\pi \in R$ that "corresponds" with (A, σ, τ).

Because $A \in \mathcal{A}$, $|A| = k$; index the elements of $A = \{a_1, a_2, \ldots, a_k\}$ so that $a_1 < a_2 < \cdots < a_k$. Similarly, index the elements of $\overline{A} = \{b_1, b_2, \ldots, b_{n-k}\}$ so that $b_1 < b_2 < \cdots < b_{n-k}$.

$$\pi_0 = \underbrace{a_1 a_2 \cdots a_k}\,\underbrace{b_1 b_2 \cdots b_{n-k}}$$

$$\downarrow \sigma \qquad\qquad \downarrow \tau$$

$$\pi = \overbrace{a_{\sigma(1)} a_{\sigma(2)} \cdots a_{\sigma(k)}}\,\overbrace{b_{\tau(1)} b_{\tau(2)} \cdots b_{\tau(n-k)}}$$

Formally, define $f((A, \sigma, \tau)) = \pi$, where

$$\pi(j) = \begin{cases} a_{\sigma(j)} & 1 \le j \le k \\ b_{\tau(j-k)} & k+1 \le j \le n. \end{cases}$$

Note that $f : \mathcal{A} \times S \times T \to R$ is a function, i.e. $f((A,\sigma,\tau)) \in R$.

We must check that f is injective and surjective.

(·) f is injective: consider $(A_1,\sigma_1,\tau_1) \neq (A_2,\sigma_2,\tau_2)$ in $\mathcal{A} \times S \times T$. Let $\pi_1 = f((A_1,\sigma_1,\tau_1))$ and $\pi_2 = f((A_2,\sigma_2,\tau_2))$. ~~If~~ ~~Last~~

Write $\pi_1 = x_1 x_2 \cdots x_n$ and $\pi_2 = y_1 y_2 \cdots y_n$.

If $A_1 \neq A_2$, then $A_1 = \{x_1,\ldots,x_k\}$ and $A_2 = \{y_1,\ldots,y_k\}$ force $\pi_1 \neq \pi_2$.
Otherwise, consider the case $A_1 = A_2$.
If $\sigma_1 \neq \sigma_2$, then π_1 and π_2 differ on some $1 \leq j \leq k$, so $\pi_1 \neq \pi_2$.

If $\tau_1 \neq \tau_2$, then π_1 and π_2 differ on some $k+1 \leq j \leq n$, so $\pi_1 \neq \pi_2$.

(·) f is surjective: if $\pi \in R$, then let A be the first k entries of π and choose $\sigma \in S$ and $\tau \in T$ so that $f((A,\sigma,\tau)) = \pi$.

Ex: · $n=8$, $k=3$, $\mathcal{U}=\{1,2,\dots,8\}$

· $\mathcal{A}=\{A\subseteq\mathcal{U} \mid |A|=3\}=\{\{1,2,3\},\{1,2,4\},\dots\{6,7,8\}\}$

· $S=\{\sigma \mid \sigma \text{ is a perm. of } [3]\}$

· $T=\{\tau \mid \tau \text{ is a perm of } [5]\}$

· $R=\{\pi \mid \pi \text{ is a perm of } [8]\}$

$f((\{2,4,5\}, 213, 54321))$:

$$\pi_0 = \underbrace{245}\ \underbrace{13678}$$

$$\downarrow\sigma \qquad \downarrow\tau$$

$$\pi = 425\ 87631$$

What about finding something that maps to

$\pi = 73412586$?

Choose $A=\{3,4,7\}$, $\sigma=312$, $\tau=12354$

$$f((A,\sigma,\tau))=\pi$$