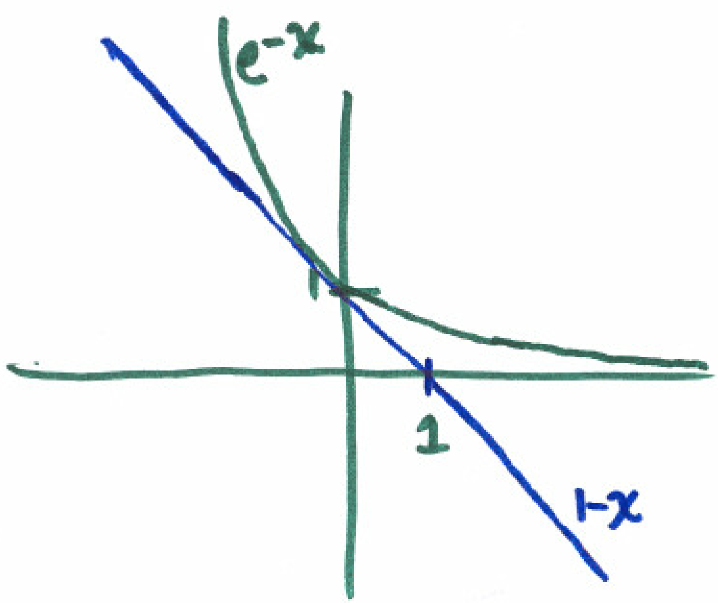
Last Time:

- We found that if we throw r balls into n bins, the probability of the event A that each ~~ball~~ bin has at most one ball is

$$\Pr(A) = \frac{n}{n} \cdot \frac{n-1}{n} \cdot \frac{n-2}{n} \cdot \ldots \cdot \frac{n-r+1}{n}$$

- To get a better understanding of what this product does, we can use the following upper bound:

$$\boxed{\forall x \quad 1-x \leq e^{-x}}$$



- Note how the closer x is to 0, the better the ~~app~~ upper bound. In fact,

$$\boxed{\forall x \geq 0 \quad e^{-x} - \frac{x^2}{2} \leq 1 - x \leq e^{-x}}$$

so in many cases, e^{-x} is a good approximation for $1-x$.

- This approximation/upper bound is useful when we are working with products such as

$$(1-x_1)(1-x_2)(1-x_3)\cdots(1-x_t) \leq e^{-x_1} \cdot e^{-x_2} \cdot \ldots \cdot e^{-x_t}$$

valid when $1 \geq x_1, x_2, \ldots, x_t \geq 0$

$$= e^{-(x_1 + x_2 + \cdots + x_t)}$$

because we can use it to convert a product to a sum.

- Therefore

$$\Pr(A) = \frac{n}{n} \cdot \frac{n-1}{n} \cdot \frac{n-2}{n} \cdot \ldots \cdot \frac{n-r+1}{n}$$

$$= \left(1 - \frac{0}{n}\right)\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \cdot \ldots \cdot \left(1 - \frac{r-1}{n}\right)$$

$$\leq e^{-0/n} \cdot e^{-1/n} \cdot e^{-2/n} \cdot \ldots \cdot e^{-r-1/n}$$

$$= e^{-\frac{1}{n}(0+1+2+\cdots+r-1)}$$

$$= e^{-\frac{1}{n}\binom{r}{2}}$$

$$= e^{-\frac{r(r-1)}{2n}}$$

- In fact, as long as r is not too large compared to n (i.e. $r = o(n^{2/3})$), this is a good approximation to $\Pr(A)$.
- This tells us that when $r = \Theta(\sqrt{n})$, there is a constant probability that some bin contains ≥ 2 balls.
- Our approximation tells us:

r	Result
$o(\sqrt{n})$	With high probability, balls fall into distinct bins
$\Theta(\sqrt{n})$	? Constant probability either way
$\omega(\sqrt{n})$	W.h.p., some bin contains ≥ 2 balls

because

$$r = o(\sqrt{n}) \implies Pr(A) \approx e^{\frac{-r(r-1)}{2n}} \longrightarrow 1$$

$$r = \Theta(\sqrt{n}) \implies Pr(A) \longrightarrow c$$

$$r = \omega(\sqrt{n}) \implies Pr(A) \longrightarrow 0$$

as $n \to \infty$.

- We say that an event A happens ~~with~~ <u>with high probability</u> if

$$Pr(A) \to 1 \quad \text{as} \quad n \to \infty.$$

Ex: For $r = n^{1/3}$, $r = n^{1/2}$, $r = n^{3/5}$ plot

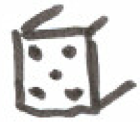
verify that

$$\lim_{n \to \infty} e^{-\frac{r(r-1)}{2n}} = \begin{cases} 1 & r = n^{1/3} \\ c & r = n^{1/2} \\ 0 & r = n^{3/5} \end{cases}$$

Recall: $[n] = \{1, 2, \dots n\}$

Method of Conditional Probabilities

<u>Question</u>: Suppose we roll a fair six-sided die n times. What is the probability that the sum of all numbers rolled is divisible by 5?

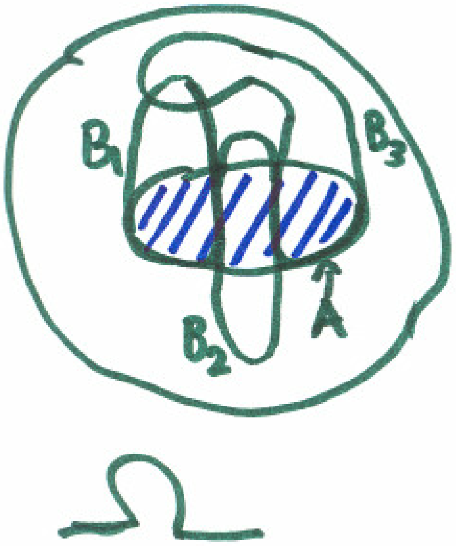
- We might be tempted to answer $1/5$. But careful! If $n=1$, the correct answer is $1/6$ (only rolling a five  works).

- It is true that this probability approaches $1/5$ as $n \to \infty$. But can we find it exactly?

- More difficult than the related problem on Exam 2.

- We need a new technique: the method of conditional probabilities.

- This techniques lets us split up a difficult probability computation into easier parts based upon several cases.



<u>Prop</u> Let A be an event and let $B_1, B_2, \ldots, B_r$ be events such that

(1) $A \subseteq B_1 \cup B_2 \cup \cdots \cup B_r$

(2) $\forall i \neq j \quad B_i \cap B_j \cap A = \emptyset$.

Then we have

$$\boxed{P_r(A) = \sum_{j=1}^{r} P_r(A \mid B_j) \cdot P_r(B_j)}$$

<u>Proof</u>: $$\sum_{j=1}^{r} P_r(A \mid B_j) \cdot P_r(B_j) = \sum_{j=1}^{r} \frac{P_r(A \cap B_j)}{P_r(B_j)} \cdot P_r(B_j)$$

$$= \sum_{j=1}^{r} P_r(A \cap B_j)$$

(A is the disjoint union $(A \cap B_1) \cup \cdots \cup (A \cap B_r)$) $= P_r(A)$ ∎

<u>Think</u>: We want to calculate $\Pr(A)$; to do so, we split up A into cases $B_1, B_2, \ldots, B_r$ which are disjoint in A. We win if we can compute $\Pr(A \mid B_j)$ and $\Pr(B_j)$.

(In fact, sometimes we don't have to compute this much...)

<u>Application</u>: Let A be the event that the sum of n rolls of a six-sided die is divisible by 5.

Note: if we were rolling a 5-sided die, we could ~~[illegible]~~ quickly conclude the answer is $1/5$.

How should we deal with the 6's?

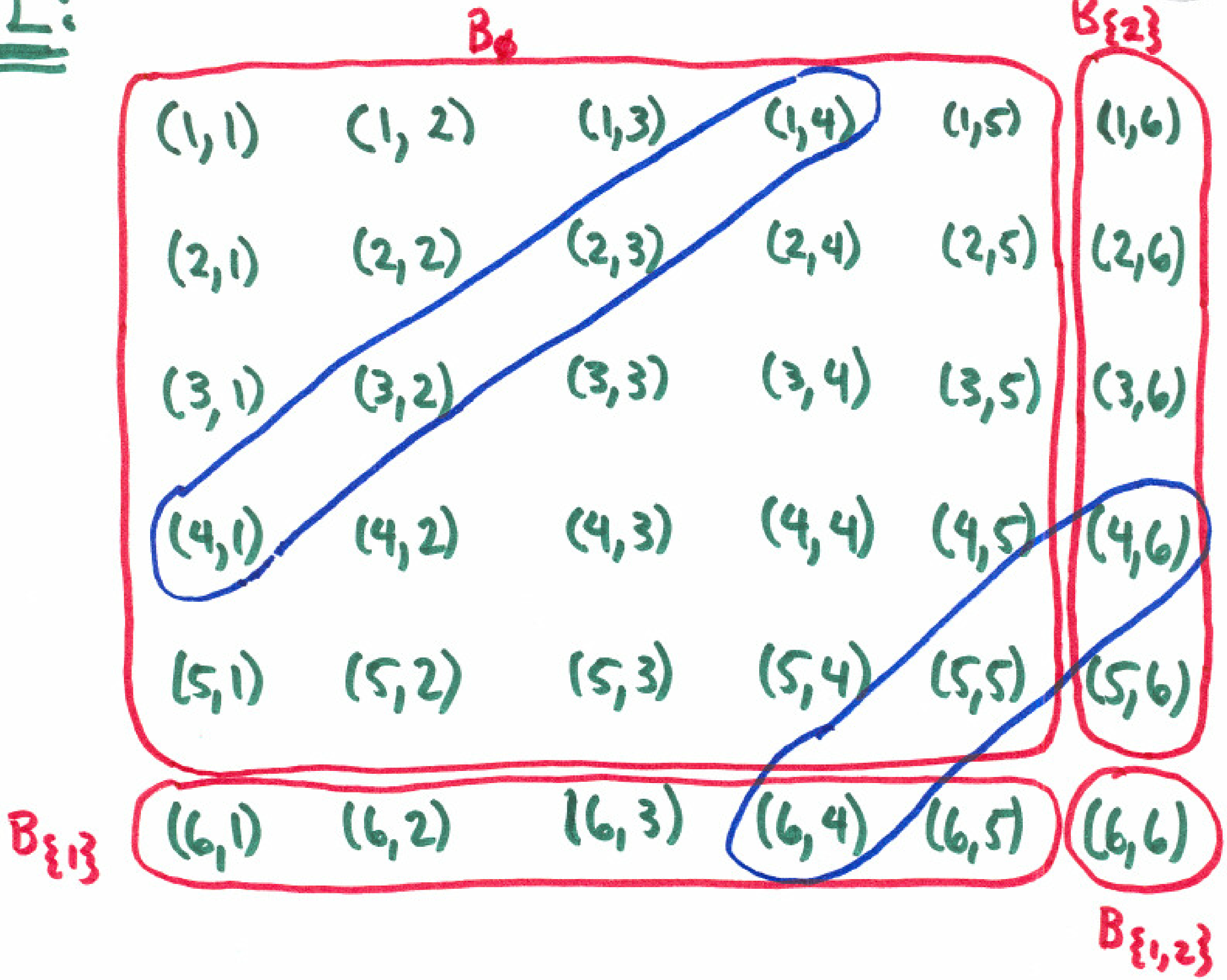
Conditionalize!

For each $S \subseteq \{1, 2, \ldots, n\}$, let B_S be the event that $\forall j$ the jth roll is a 6 iff $j \in S$.

Ex

- $n=2$
- $\Omega = \{1,2,\dots,6\} \times \{1,2,\dots,6\} = \{(1,1),(1,2),\dots,(6,6)\}$
- We define
  - $B_\emptyset = \{(1,1),(1,2),\dots,(1,5),$
    $(2,1),(2,2),\dots,(2,5),$
    $\vdots$
    $(5,1),(5,2),\dots,(5,5)\} = \{1,2,\dots,5\} \times \{1,\dots,5\}$
  - $B_{\{1\}} = \{(6,1),(6,2),\dots,(6,5)\}$
  - $B_{\{2\}} = \{(1,6),(2,6),\dots,(5,6)\}$
  - $B_{\{1,2\}} = \{(6,6)\}$

$\underline{\underline{\Omega}}$:



- Picture of our sample space Ω, the sets $B_\emptyset$, $B_{\{1\}}$, $B_{\{2\}}$, $B_{\{1,2\}}$, and event A circled in blue

- Note that the sample space is the disjoint union

$$\Omega = \bigcup_{S \subseteq \{1,\dots,n\}} B_S$$

so $A \subseteq \Omega = \bigcup_S B_S$ and the events are disjoint in A.

- Therefore

$$P_r(A) = \sum_{S \subseteq \{1,\dots,n\}} P_r(A \mid B_S) \cdot P_r(B_S)$$

$$= \left(\sum_{\substack{S \subseteq [n] \\ S \neq \{1,2,\dots,n\}}} P_r(A \mid B_S) \cdot P_r(B_S) \right)$$

$$+ P_r(A \mid B_{\{1,2,\dots,n\}}) \cdot P_r(B_{\{1,2,\dots,n\}})$$

- First, let's start with the last term. Because $B_{\{1,2,\ldots,n\}}$ is the event that all n rolls are 6 and the rolls are independent, we get $\Pr(B_{\{1,\ldots,n\}}) = \left(\frac{1}{6}\right)^n$

- What is $\Pr(A \mid B_{\{1,2,\ldots,n\}})$? Well, under the assumption that $B_{\{1,2,\ldots,n\}}$ occured, i.e. that we've rolled n 6's, our sum is $6n$ which is divisible by $5 \Leftrightarrow n$ is divisible by 5. Therefore

$$\Pr(A \mid B_{\{1,\ldots,n\}}) = \begin{cases} 1 & n \text{ div. by } 5 \\ 0 & \text{otherwise} \end{cases}$$

- Suppose $S \subseteq \{1,2,\ldots,n\}$ but $S \neq \{1,\ldots,n\}$. What is $\Pr(A \mid B_S)$? Well, now that we've conditionalized on which rolls came up

(10.1)

$$B_{\{1,2,\ldots,n\}} = (\text{1st roll is } 6) \cap (\text{2nd roll is } 6) \cap \cdots \cap (n\text{th roll is } 6)$$

Because these events are mutually independent

$$\Rightarrow \Pr\left(B_{\{1,2,\ldots,n\}}\right) = \Pr(\text{1st roll is } 6) \cdot \Pr(\text{2nd roll is } 6) \cdot \ldots \cdot \Pr(n^{\text{th}} \text{ roll is } 6)$$

$$= \frac{1}{6} \cdot \frac{1}{6} \cdot \ldots \cdot \frac{1}{6} = \left(\frac{1}{6}\right)^n$$

Ex: $S = \{1, 2, 4\}$, $n = 6$

$$\Omega = \{(x_1, x_2, x_3, x_4, x_5, x_6) \mid 1 \le x_i \le 6\}$$

$$B_S = \{(\underline{6}, \underline{6}, \underline{x_3}, \underline{6}, \underline{x_5}, \underline{x_6}) \mid 1 \le x_3, x_5, x_6 \le 5\}$$

6, we are asking for the probability that ~~the~~ rolling a 5-sided die $n - |S|$ times results in a sum that is $-6|S| \equiv -|S| \mod 5$.

(The notation $x \bmod 5$ means to take the remainder you get when you divide x by 5:

$$1 \bmod 5 = 1$$
$$12 \bmod 5 = 2$$
$$-3 \bmod 5 = 2.$$

If you are unfamiliar with this, Google for "Modular arithmetic".)

Because $S \neq \{1, \ldots, n\}$, we are rolling our 5-sided die $n - |S| \geq 1$ times, and each class (mod 5) is equally likely to be the sum. Therefore $\Pr(A \mid B_S) = \frac{1}{5}$.

Therefore

$$Pr(A) = \left(\sum_{\substack{S \subseteq \{1,2,\ldots,n\} \\ S \neq \cancel{\emptyset} \{1,2,\ldots,n\}}} Pr(A|B_S) \cdot Pr(B_S) \right) + Pr(A|B_{[n]}) \cdot Pr(B_{[n]})$$

$$= \left(\sum_{\substack{S \subseteq [n] \\ S \neq \cancel{\emptyset}[n]}} \frac{1}{5} \cdot Pr(B_S) \right) + \left(\frac{1}{6}\right)^n \cdot \begin{cases} 1 & n \equiv 0 \bmod 5 \\ 0 & \text{otherwise} \end{cases}$$

$$= \frac{1}{5} \cdot \left(1 - Pr(B_{[n]})\right) + \left(\frac{1}{6}\right)^n \cdot \begin{cases} 1 & n \equiv 0 \bmod 5 \\ 0 & \text{otherwise} \end{cases}$$

$$= \frac{1}{5} \cdot \left(1 - \left(\frac{1}{6}\right)^n\right) + \left(\frac{1}{6}\right)^n \cdot \begin{cases} 1 & n \equiv 0 \bmod 5 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{1}{5}\left(1 + \frac{4}{6^n}\right) & n \equiv 0 \bmod 5 \\ \frac{1}{5}\left(1 - \frac{1}{6^n}\right) & \text{otherwise} \end{cases}$$