Discrete Probability

- Probability theory models situations where the outcome of an experiment is <u>uncertain</u> and subject to random chance.

- <u>Ex</u>: Roll a pair of dice. We do not know before the dice are rolled what numbers will show up on the face of the dice.

⇒ Flip a coin. We'll either get a "Heads" or a "Tails".

def A probability space consists of three objects:

(1) A sample space Ω which is a non-empty set containing all the possible outcomes of the experiment

(2) A collection of events; each event is a subset of Ω

(3) A function Pr ~~from~~ which maps each event $A \subseteq \Omega$ to its probability $Pr(A)$.

The probability function ~~satisfies~~ has 3 properties:

(·) For each event A, $0 \leq Pr(A) \leq 1$.

(·) $Pr(\Omega) = 1$.

(·) If $A_1, A_2, A_3, \cdots$ are pairwise disjoint events and $A = A_1 \cup A_2 \cup A_3 \cup \cdots = \bigcup_{j=1}^{\infty} A_j$, then

$$\Pr(A) = \Pr(A_1) + \Pr(A_2) + \Pr(A_3) + \cdots$$
$$= \sum_{j=1}^{\infty} \Pr(A_j) .$$

[redacted]

«Insert Visualization Slide Here»

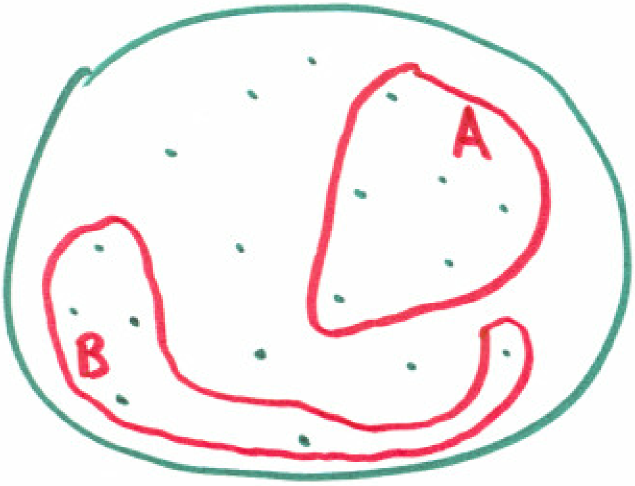
Ex: We can use a probability space to model flipping two [redacted] "fair" coins:

- $\Omega = \{HH, HT, TH, TT\}$

-

Event	$\Pr(A)$	Event	$\Pr(A)$
$\{\}$	0	$\{HT, TH\}$	$1/2$
$\{HH\}$	$1/4$	$\{HT, TT\}$	$1/2$
$\{HT\}$	$1/4$	$\{TH, TT\}$	$1/2$
$\{TH\}$	$1/4$	$\{HH, HT, TH\}$	$3/4$
$\{TT\}$	$1/4$	$\{HH, HT, TT\}$	$3/4$
$\{HH, HT\}$	$1/2$	$\{HH, TH, TT\}$	$3/4$
$\{HH, TH\}$	$1/2$	$\{HT, TH, TT\}$	$3/4$
$\{HH, TT\}$	$1/2$	$\{HH, HT, TH, TT\}$	1

Visualization: If A, B are disjoint:



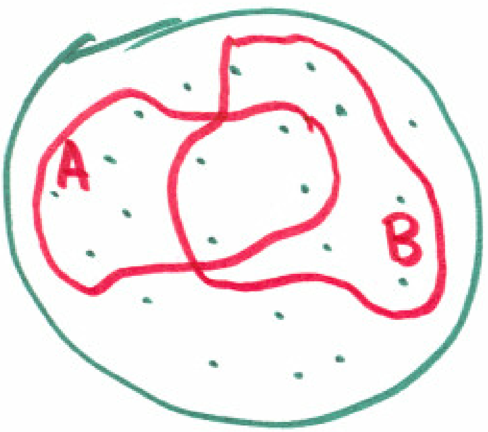
Ω

- $\Pr(A \cup B) = \Pr(A) + \Pr(B)$
- $\Pr(\Omega) = \Pr(A \cup \bar{A})$

  $= \Pr(A) + \Pr(\bar{A})$,

  so $\boxed{\begin{aligned}\Pr(\bar{A}) &= \Pr(\Omega) - \Pr(A) \\ &= 1 - \Pr(A)\end{aligned}}$

- In general:



Ω

Thm $\boxed{\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)}$

Pf:

- $\Pr(A \cup B) = \Pr(A-B) + \Pr(A \cap B) + \Pr(B-A)$

$= \left(\Pr(A-B) + \Pr(A \cap B)\right) + \left(\Pr(B-A) + \Pr(A \cap B)\right) - \Pr(A \cap B)$

$= \Pr(A) + \Pr(B) - \Pr(A \cap B)$

- Lots of times we use English to define an event.

"Let A be the event that the second flip comes up heads" means "Let $A = \{HH, TH\}$"

"Let B be the event that the ~~both~~ the coins ~~flips~~ land with the same side face up" means "Let $B = \{HH, TT\}$."

<u>Remark</u>:

- In many cases, our sample space Ω only contains a finite number of outcomes
- If each outcome is equally likely (i.e. $\omega_i, \omega_j \in \Omega \Rightarrow P_r(\{\omega_i\}) = P_r(\{\omega_j\})$)

then

$$|\Omega| \cdot P_r(\{\omega_i\}) = \sum_{\omega \in \Omega} P_r(\{\omega\}) = P_r(\Omega) = 1$$

so for each $\omega_i \in \Omega$,

$$Pr(\{\omega_i\}) = \frac{1}{|\Omega|}$$

- In general, if ~~[illegible]~~ Ω is finite,

$$Pr(A) = \sum_{\omega \in A} Pr(\{\omega\})$$

If each outcome in Ω is equally likely, this simplifies:

$$\boxed{\begin{aligned} Pr(A) &= \sum_{\omega \in A} Pr(\{\omega\}) \\ &= \sum_{\omega \in A} \frac{1}{|\Omega|} \\ &= \frac{|A|}{|\Omega|} \end{aligned}}$$

Ex: Suppose we roll two six-sided dice. Our sample space is $\Omega = \{1,2,\ldots,6\} \times \{1,2,\ldots,6\}$ so that $(r,s) \in \Omega$ denotes the outcome of rolling $1 \le r \le 6$ on the first die and rolling $1 \le s \le 6$ on the second die.

Each outcome is equally likely, so

$$\Pr(A) = \frac{|A|}{|\Omega|} = \frac{|A|}{36}.$$

Let A_n be the event that the numbers on the dice sum to n.

r \ s	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Table shows $r+s$

(·) $A_1 = \emptyset$

$A_2 = \{(1,1)\}$

$A_3 = \{(1,2), (2,1)\}$

$A_4 = \{(1,3), (2,2), (3,1)\}$

n	2	3	4	5	6	7	8	9	10	11	12
$Pr(A_n)$	$1/36$	$2/36$	$3/36$	$4/36$	$5/36$	$6/36$	$5/36$	$4/36$	$3/36$	$2/36$	$1/36$

So, the most likely sum is 7 with

$$Pr(A_7) = \frac{6}{36}.$$

<u>Exercise:</u> Which number is the most likely product?

Ex: Let $n \geq 1$ be an integer, and let Ω be the sample space $\{H,T\}^n$ that represents flipping a coin n times. We assume the coin is fair, so each sequence $\omega \in \Omega$ of coin flips is equally likely.

(·) What is the probability that all coin flips give the same result?

Soln: Let A be the event that all flips are the same. Note

$$A = \{HHH\cdots H, TTT\cdots T\}$$

and

$$\Pr(A) = \frac{|A|}{|\Omega|} = \frac{2}{2^n} = \frac{1}{2^{n-1}}.$$

(·) What is the probability that at least one head and at least one tail appears?

Soln: Let B be the event that at least one head and at least one tail appears. Note that $\overline{B}$ is the event that either

(·) No head appears (all tails flipped), or

(·) No tail appears (all heads flipped),

so $\overline{B} = A$.

Therefore
$$\begin{aligned} \Pr(B) &= 1 - \Pr(\overline{B}) \\ &= 1 - \Pr(A) \\ &= 1 - \frac{1}{2^{n-1}}. \end{aligned}$$

(·) Suppose n is even. What is the probability that the same number of heads and tails are flipped?

Soln: Let $A = \{\omega \in \Omega \mid \omega \text{ contains } n/2 \text{ heads}\}$.

Note that $|A| = \binom{n}{n/2}$, and so

$$\Pr(A) = \frac{\binom{n}{n/2}}{2^n} \approx \frac{1}{\sqrt{n}}$$

(·) Suppose $n \geq 2$. What is the probability that ~~the first and~~ we flip a heads within the first two coin flips?

Soln: Let $A = \{ \omega \mid \text{first flip is heads} \}$,

$B = \{ \omega \mid \text{second flip is heads} \}$.

Check that $|A| = 2^{n-1}$

$|B| = 2^{n-1}$

$|A \cap B| = 2^{n-2}$

and therefore

$$\Pr(A) = \Pr(B) = \frac{|A|}{|\Omega|} = \frac{2^{n-1}}{2^n} = \frac{1}{2}$$

$$\Pr(A \cap B) = \frac{|A \cap B|}{|\Omega|} = \frac{2^{n-2}}{2^n} = \frac{1}{4}$$

It follows that

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$

$$= \frac{1}{2} + \frac{1}{2} - \frac{1}{4}$$

$$= 3/4$$

(·) What is the probability that we do not flip heads twice in a row?

<u>Soln</u>: This one is not as simple. ~~Let~~ It is difficult to directly count the number of sequences of n coin flips without two consecutive heads.

Let A be the event that we don't flip 2 heads in a row.

<u>Things to check</u>:

(·) Is the complement $\bar{A}$ = sequences where we <u>do</u> flip 2 ~~coins~~ heads in a row easier to count?

Not really...

(·) Can we express A as the union of disjoint events, compute their probabilities, and sum them up?

Also not so easy.

We need a new trick:

Recurrence Relations to the rescue!

Let $T(n)$* be the number of sequences of n coin flips in which there are no consecutive heads.

How can such a sequence begin? There are two cases:

Case 1: First flip is a tail. If $w = T\underbrace{\cdots\cdots}_{n-1 \text{ flips}}$, then w does not contain 2 heads in a row iff the last $n-1$ flips also have this property. There are $T(n-1)$ of these sequences.

Case 2: First flip is a heads. If this is the case, the second flip must be tails, so $w = HT\underbrace{\cdots\cdots}_{n-2 \text{ flips}}$ and now the last $n-2$ flips must not contain 2 heads in a row. There are $T(n-2)$ of these cases.

Because each sequence of n flips without consecutive heads falls into exactly one of these two cases, we have

$$\forall n \geq 2 \quad T(n) = T(n-1) + T(n-2)$$

So $T(n)$ satisfies the Fibonacci recurrence.

What about the base cases? For $n \in \{0,1\}$ we directly compute

$T(0) = 1$ (The empty sequence)

$T(1) = 2$ (Both H and T work.)

In Lecture 16, we solved this recurrence with different base cases: $T'(0) = T'(1) = 1$. Fortunately, $T'(2) = 2$, so our base cases above only shift the sequence and $T(n) = T'(n+1)$. Using our formula in Lecture 16, we get

$$T(n) = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+2}$$
$$\approx \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+2}$$

and so

$$\Pr(A) = \frac{|A|}{|\Omega|} = \frac{T(n)}{2^n}$$

$$\approx \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{2} \cdot \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n}}{2^n}$$

$$\approx 1.17082 \cdot (0.809)^n$$

$$= \Theta\left(\left(\frac{1+\sqrt{5}}{4}\right)^n\right)$$