Characteristic Equation Method

- Consider the **Fibonacci sequence**:

$$T(n) = \begin{cases} 1 & n=0,1 \\ T(n-1)+T(n-2) & n \geq 2 \end{cases}$$

- Exercise: prove by induction that

$$T(n-1) \leq T(n)$$

- Bounds:
  - $T(n) \leq 2 \cdot T(n-1)$; can show by induction $T(n) \leq 2^n$.
  - $T(n) \geq 2 \cdot T(n-2)$; can show by induction $T(n) \geq 2^{\frac{n-1}{2}}$.

- So $2^{\frac{n-1}{2}} = \frac{1}{\sqrt{2}}(\sqrt{2})^n \leq T(n) \leq 2^n$.

$\sqrt{2} \approx 1.414\ldots$

- Reasonable guess:

$$T(n) = r^n$$

for some $\sqrt{2} < r < 2$. (We shall see $r \approx 1.618\cdots$).

- Check the inductive step. Later, we will fix the base cases.

- Assume by I.H. that $T(n-1) = r^{n-1}$, $T(n-2) = r^{n-2}$.

- <u>Want</u>:

$$r^n \overset{\text{want}}{=} T(n) \overset{\text{know by defn}}{=} T(n-1) + T(n-2)$$

(by I.H.) $= r^{n-1} + r^{n-2}$ or $r^n - r^{n-1} - r^{n-2} = 0$

Factor r^{n-2}:

$$r^{n-2}(r^2 - r - 1) = 0$$

- So we must pick $r = 0$ or r to be a root of <u>the characteristic equation</u>

$$r^2 - r - 1 = 0$$

if $T(n) = r^n$ is to be a solution.

- $r = 0$ leads to the solution $T(n) \equiv 0$ which is of no help.
- Recall the quadratic ~~equation~~ formula:

$$ax^2 + bx + c = 0 \implies x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

- In our case,

$$r^2 - r - 1 = 0 \implies r = \frac{1 \pm \sqrt{5}}{2}$$

So

$$r \in \left\{ \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2} \right\} = \left\{ 1.618\ldots, -0.618\ldots \right\}$$

- The number $\frac{1+\sqrt{5}}{2}$ is called the golden ratio and has many special properties.

- Let $r_1 = \frac{1+\sqrt{5}}{2}$, $r_2 = \frac{1-\sqrt{5}}{2}$.

- Both $T(n) = r_1^n$ and $T(n) = r_2^n$ will solve the inductive step in the "check" portion of guess and check.

- What about the base cases?

- Note: for all numbers α, β:

$$T(n) = \alpha r_1^n + \beta r_2^n$$

will also work:

$$T(n) = T(n-1) + T(n-2)$$

(By I.H.) $= \alpha r_1^{n-1} + \beta r_2^{n-1} + \alpha r_1^{n-2} + \beta r_2^{n-2}$

(Factor α, β) $= \alpha(r_1^{n-1} + r_1^{n-2}) + \beta(r_2^{n-1} + r_2^{n-2})$

($T(n) = r_1^n, r_2^n$ are solns) $= \alpha r_1^n + \beta r_2^n$ ✓

- We can use the freedom to choose α, β however we like to satisfy the base cases. In fact,

$$T(n) = \alpha r_1^n + \beta r_2^n$$

is the general solution to the recurrence because no matter what $T(0), T(1)$ are, we can choose α, β to accommodate $T(0), T(1)$.

- In our case,

$$n=0 \Rightarrow \quad 1 = T(0) = \alpha + \beta$$

$$n=1 \Rightarrow \quad 1 = T(1) = \alpha r_1 + \beta r_2$$

$$= \alpha \cdot \frac{1+\sqrt{5}}{2} + \beta \frac{1-\sqrt{5}}{2}$$

- After algebra, we find

$$\alpha = \frac{1}{\sqrt{5}} r_1 \quad , \quad \beta = -\frac{1}{\sqrt{5}} r_2$$

and therefore

$$\boxed{T(n) = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}}$$

· Guess $T(n) = r^n$

$T(n) = 6\,T(n-1) - 9\,T(n-2)$

$= 6\,r^{n-1} - 9 \cdot r^{n-2}$

Want $r^n = 6\,r^{n-1} - 9\,r^{n-2}$

$r^2 = 6r - 9$

<u>Char. Eq.</u>: $r^2 - 6r + 9 = 0$

- Recurrences of the form

$$T(n) = c_1 T(n-1) + c_2 T(n-2) + \cdots + c_k T(n-k) + f(n)$$

are called <u>linear recurrences</u> of <u>order</u> k.

- If $f(n) \equiv 0$, then $T(n)$ is <u>homogeneous</u>.
- Otherwise, $T(n)$ is called <u>inhomogeneous</u> and $f(n)$ is the <u>inhomogeneous term</u>.

$\Rightarrow$ The techniques we've seen apply to any homogeneous linear recurrence of order 2.

<u>One complication</u>: $T(n) = 6T(n-1) - 9T(n-2)$.

In this case, the characteristic equation is

$$r^2 - 6r + 9 = 0$$

which factors as

$$(r-3)^2 = 0$$

- This tells us that $\forall \alpha$,

$$T(n) = \alpha \cdot 3^n$$

is a solution. But we cannot accommodate two base case constraints with only one variable α.

- In the case that r is a ~~solution to~~ root of the characteristic equation of multiplicity two, then $T(n) = n \cdot r^n$ is also a solution:

<u>Ex</u>:

$$T(n) = 6T(n-1) - 9T(n-2)$$

(by I.H.)

$$= 6(n-1)\cdot 3^{n-1} - 9(n-2)\cdot 3^{n-2}$$

$$= 2(n-1)\cdot 3^n - (n-2)\cdot 3^n$$

$$= [2(n-1) - (n-2)]\cdot 3^n$$

$$= n \cdot 3^n \qquad \checkmark$$

- In this case, the general solution is

$$T(n) = \alpha r^n + \beta n r^n$$

which has two variables/ two degrees of freedom and can accommodate two base cases.

In our example,

$$T(n) = \alpha \cdot 3^n + \beta \cdot n \cdot 3^n$$

is the general solution.

Inhomogeneous Recurrences

Ex: $T(n) = 5T(n-1) - 6T(n-2) + n^2$

- The corresponding homogeneous recurrence is

$$S(n) = 5 \cdot S(n-1) - 6 \cdot S(n-2).$$

First, solve for S:

$\Rightarrow$ characteristic equation: $r^2 = 5r - 6$

$$r^2 - 5r + 6 = 0$$

$$(r-2)(r-3) = 0$$

so $S(n) = 2^n$, $S(n) = 3^n$ are solutions to S and the general solution is

$$S(n) = \alpha \cdot 2^n + \beta \cdot 3^n.$$

- Next, we want to find a particular solution $T(n) = g(n)$ to T and our general solution will be

$$T(n) = \underbrace{\alpha \cdot 2^n + \beta \cdot 3^n}_{\text{general solution to } S} + g(n)$$

· Let us check this works. Let

$$h(n) = \alpha \cdot 2^n + \beta \cdot 3^n = \text{general soln to } S.$$

Now

$$T(n) = 5T(n-1) - 6T(n-2) + n^2$$

(by I.H.) $= 5(h(n-1) + g(n-1)) - 6(h(n-2) + g(n-2)) + n^2$

(group terms) $= 5h(n-1) - 6h(n-2)$

$+ 5g(n-1) - 6g(n-2) + n^2$

(h is soln to S, g is soln to T) $= h(n) + g(n)$ ✓

- How do we find a particular solution

$$T(n) = g(n) \ ?$$

- Suppose that the inhomogeneous term is a polynomial.
- The trick is to "guess" a polynomial.

The degree of the guessed polynomial should be the same as the degree of the inhomogeneous term.

$$T(n) = 5T(n-1) - 6T(n-2) + \underbrace{n^2}_{\text{degree } 2}$$

- <u>Guess</u>: $T(n) = an^2 + bn + c$. In order for our guess to work, we must have

$$an^2 + bn + c = 5\left[a(n-1)^2 + b(n-1) + c\right] - 6\left[a(n-2)^2 + b(n-2) + c\right] + n^2$$

- Multiply on R.H.S. and collect terms:

$$an^2 + bn + c = (-a+1)n^2 + (14a-b)n + (-19a+7b-c)$$

- Set coefficients equal

n^2: $a = -a + 1 \Rightarrow a = 1/2$

n: $b = 14a - b \Rightarrow 7 - b \Rightarrow b = 7/2$

n^0: $c = -19a + 7b - c = \frac{-19}{2} + \frac{49}{2} - c$

$$\Rightarrow 2c = 15 \Rightarrow c = 15/2.$$

So our particular solution is

$$T(n) = 1/2\, n^2 + 7/2\, n + 15/2$$

and our general solution is

$$T(n) = \alpha \cdot 2^n + \beta \cdot 3^n + 1/2\, n^2 + 7/2\, n + 15/2.$$

- If we were given base cases, we would now write down equations

$n=0 \Rightarrow \quad T(0) = \alpha + \beta + 15/2$

$n=1 \Rightarrow \quad T(1) = 2\alpha + 3\beta + 23/2$

and solve the system; find α, β in terms of $T(0), T(1)$.

<u>Remarks</u>: You now have the tools needed to solve must recurrences you'll see in CS473.

- Prof Jeff Erickson has ~~a~~ notes on solving recurrences:

<u>Google</u>: $\boxed{\text{"Jeff Erickson" recurrence notes}}$