Little-oh notation

<u>Recall</u>: $f, g : \{1, 2, \ldots\} \to \mathbb{R}^+$, $g(n) > 0$.

- $f(n) = O(g(n)) \iff \exists c$ for all sufficiently large n, $\frac{f(n)}{g(n)} \leq c$
- $f(n) = \Omega(g(n)) \iff \exists c > 0$ for all sufficiently large n, $\frac{f(n)}{g(n)} \geq c$

<u>def</u> $\boxed{f(n) = o(g(n))}$ if $\lim_{n \to \infty} \frac{f(n)}{g(n)} =$ zero

$\boxed{f(n) = \omega(g(n))}$ if $\lim_{n \to \infty} \frac{f(n)}{g(n)} = \infty$

<u>Exercise</u>: Prove that

(1) $f(n) = o(g(n)) \implies f(n) = O(g(n))$ but not $f(n) = \Omega(g(n))$

(2) $f(n) = \omega(g(n)) \implies f(n) = \Omega(g(n))$ but not $f(n) = O(g(n))$

Ex: • $f(n) = \sqrt{n}$, $g(n) = n$; then $f(n) = o(g(n))$ because

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{\sqrt{n}}{n} = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$

• $f(n) = n^{28}$, $g(n) = 2^n$; then $f(n) = o(g(n))$ because

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{n^{28}}{2^n} = 0$$

Ex: $f(n) = 2^n, \quad g(n) = n! \qquad f(n) = o(g(n))$

$$\lim_{n\to\infty} \frac{2^n}{n!} = \lim_{n\to\infty} \frac{2\cdot 2\cdot 2 \cdots 2}{n\cdot n-1\cdot n-2 \cdots 1}$$

$$= \lim_{n\to\infty} \frac{2}{n}\cdot\frac{2}{n-1}\cdot\frac{2}{n-2}\cdots\frac{2}{2}\cdot\frac{2}{1}$$

$$\leq \lim_{n\to\infty} \frac{2}{n}\cdot 1\cdot 1 \cdots 1\cdot 2$$

$$\leq \lim_{n\to\infty} \frac{4}{n} = 0$$

Think:

$f(n) = o(g(n))$ means $f < g$

$f(n) = O(g(n))$ means $f \leq g$

$f(n) = \Omega(g(n))$ means $f \geq g$

$f(n) = \omega(g(n))$ means $f > g$

$f(n) = \Theta(g(n))$ means $f \approx g$

But careful:

Exercise: Find a pair of functions f, g such that

(·) $f(n) = O(g(n))$

(·) $f(n) = \Omega(g(n))$ does not hold

(·) $f(n) = o(g(n))$ does not hold

~~Ex~~ Solution to Exercise:

- $g(n) = n$
- $f(n) = \begin{cases} n & n \text{ is even} \\ 1 & n \text{ is odd.} \end{cases}$

$$\frac{f(n)}{g(n)} = \begin{cases} 1 & n \text{ is even} \\ 1/n & n \text{ is odd} \end{cases}$$

- $\frac{f(n)}{g(n)} \leq 1 \implies f(n) = O(g(n))$
- Because $\lim_{n \to \infty} \frac{f(n)}{g(n)}$ is not defined, $f(n) = o(g(n))$ does not hold.
- Also, $f(n) = \Omega(g(n))$ does not hold.

A tale of two sums

(1) Let α be some number. A surprisingly common sum, ~~is~~ known as a geometric series, is

$$S = 1 + \alpha + \alpha^2 + \alpha^3 + \cdots + \alpha^n$$
$$= \sum_{j=0}^{n} \alpha^j$$

Here is the trick to find the formula for S:

$$S = 1 + \alpha + \alpha^2 + \alpha^3 + \cdots + \alpha^{n-1} + \alpha^n$$
$$\alpha \cdot S = \alpha + \alpha^2 + \alpha^3 + \cdots + \alpha^{n-1} + \alpha^n + \alpha^{n+1}$$

So

$$S - \alpha \cdot S = 1 - 0 - 0 - 0 - 0 - 0 - \alpha^{n+1}$$
$$= 1 - \alpha^{n+1}$$

And then $S(1-\alpha) = 1 - \alpha^{n+1}$, so

$$\boxed{S = \frac{1-\alpha^{n+1}}{1-\alpha}}$$

Also, if $|\alpha| < 1$, then

$$1 + \alpha + \alpha^2 + \alpha^3 + \cdots = \sum_{j=0}^{\infty} \alpha^j$$

$$= \lim_{n\to\infty} \sum_{j=0}^{n} \alpha^j$$

$$= \lim_{n\to\infty} \frac{1-\alpha^{n+1}}{1-\alpha}$$

$$= \frac{1}{1-\alpha} \cdot \left(\lim_{n\to\infty} 1 - \alpha^{n+1} \right)$$

$$= \boxed{\frac{1}{1-\alpha}}$$

Ex: [redacted]

$$\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \cdots = \left(\sum_{j=0}^{\infty} \left(-\frac{1}{2}\right)^j \right) \cdot 1/2$$

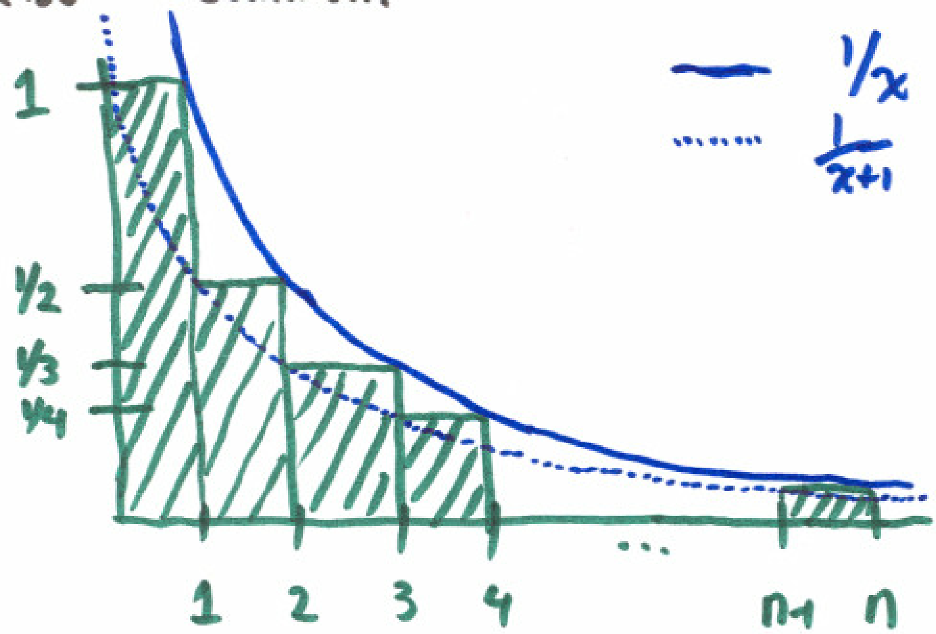
$$= \frac{1}{1-(-\frac{1}{2})} \cdot \frac{1}{2} = \frac{1}{3/2} \cdot \frac{1}{2}$$

$$= \frac{1}{3}$$

(2) The harmonic series, defined

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

is also common.



$$\ln n < \ln(n+1) = \ln(x+1)\Big|_0^n = \int_0^n \frac{1}{x+1}\,dx \le H_n$$

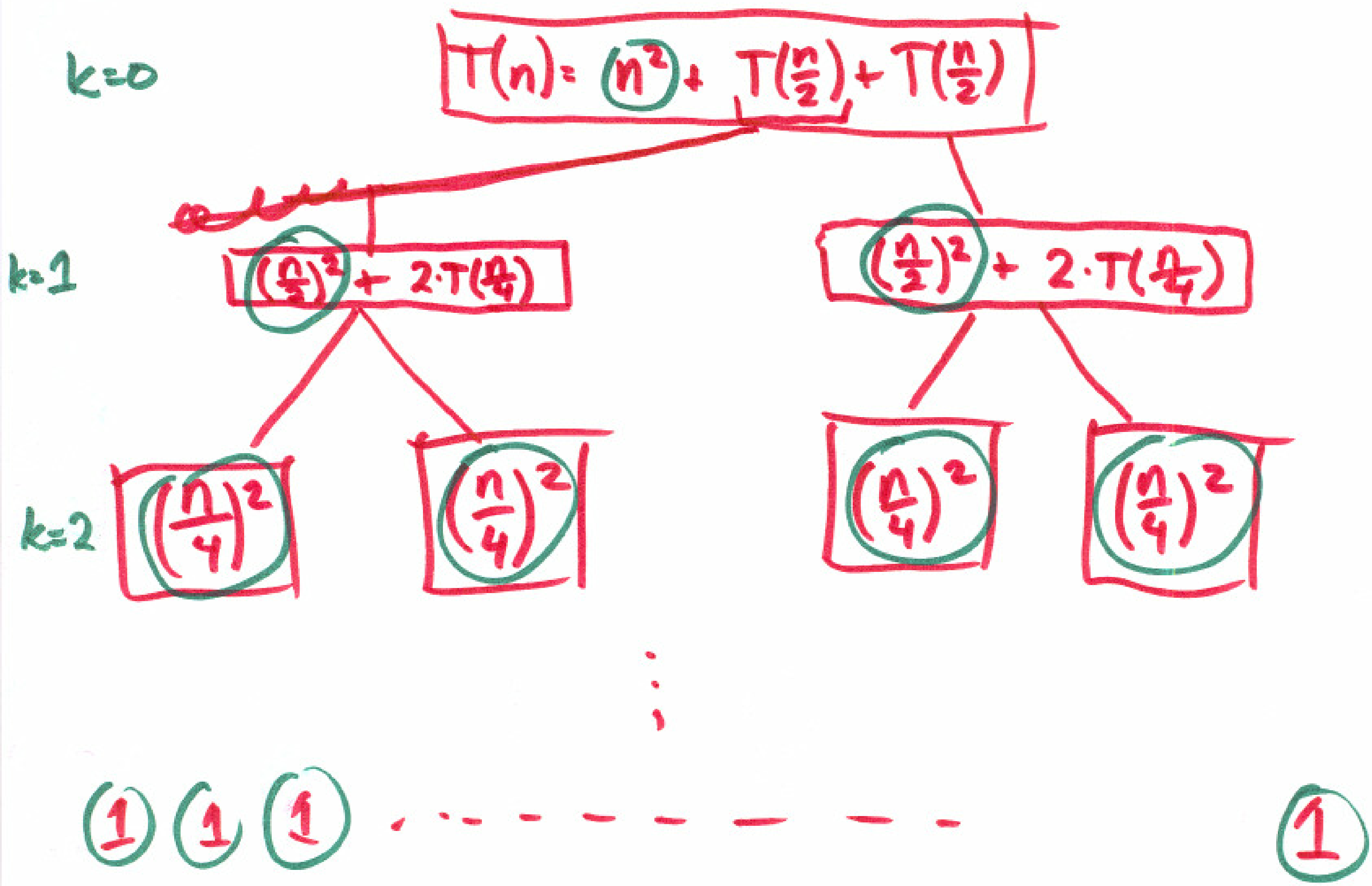
$$H_n \le 1 + \int_1^n \frac{1}{x}\,dx = 1 + \ln n$$

Therefore:

$$\boxed{\begin{array}{c} \ln n < H_n \le \ln n + 1 \\ H_n \approx \ln n \end{array}}$$ and $H_n = \Theta(\log n)$

Ex: $T(n) = 2\cdot T(\frac{n}{2}) + n^2$

- Assume a reasonable base case ($T(1) = 1$)



depth	contribution/node	# nodes	total at depth
0	n^2	1	n^2
1	$\left(\frac{n}{2}\right)^2$	2	$2 \cdot \left(\frac{n}{2}\right)^2$
2	$\left(\frac{n}{4}\right)^2$	4	$4 \cdot \left(\frac{n}{4}\right)^2$
$\vdots$			
k	$\left(\frac{n}{2^k}\right)^2$	2^k	$2^k \cdot \left(\frac{n}{2^k}\right)^2 = \frac{1}{2^k} \cdot n^2$
$\vdots$			
$d = \lg n$	1	$2^d = n$	n

So, summing the contributions, we get

$$T(n) = \sum_{k=0}^{d} \frac{1}{2^k} \cdot n^2$$

$$= n^2\left(1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^d}\right)$$

So $n^2 \le T(n) \le 2n^2$ because

$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 2$ and $T(n) = \Theta(n^2)$.

Ex $T(n) = 3\cdot T(n/3) + \frac{n}{\lg n}$

· Assume base case $T(1) = 0$.

$$T(n) = \boxed{3\cdot T(n/3) + \frac{n}{\lg n}}$$

k=0

$\boxed{3\cdot T(\frac{n}{9}) + \frac{n/3}{\lg n/3}}$ $\boxed{3\cdot T(\frac{n}{9}) + \frac{n/3}{\lg n/3}}$ $\boxed{3\cdot T(\frac{n}{9}) + \frac{n/3}{\lg n/3}}$

k=1

$\vdots$

$$3\cdot T\left(\frac{n}{3^{k+1}}\right) + \frac{n/3^k}{\lg n/3^k}$$

k

depth	argument	work/node	total nodes	total work
0	n	$\frac{n}{\lg n}$	1	$\frac{n}{\lg n}$
1	$n/3$	$\frac{n/3}{\lg n/3}$	3	$\frac{n}{\lg n/3}$
2	$n/9$	$\frac{n/9}{\lg n/9}$	9	$\frac{n}{\lg n/9}$
$\vdots$				
k	$n/3^k$	$\frac{n/3^k}{\lg n/3^k}$	3^k	$\frac{n}{\lg n/3^k}$
$\vdots$				
d	1	[illegible]	3^d	0

The total depth of the recursion tree is

$$d = \log_3 n.$$

We get

$$T(n) = \sum_{k=0}^{d-1} \frac{n}{\lg n/3^k} = n \sum_{k=0}^{d-1} \frac{1}{\lg n - k \cdot \lg 3}$$

$$= \frac{n}{\lg 3} \sum_{k=0}^{d-1} \frac{1}{\log_3 n - k}$$

$$= \frac{n}{\lg 3} \sum_{k=0}^{d-1} \frac{1}{d-k}$$

$$= \frac{n}{\lg 3} \cancel{2}\left(\frac{1}{d} + \frac{1}{d-1} + \frac{1}{d-2} + \cdots \frac{1}{1}\right)$$

$$= \frac{n}{\lg 3} \cdot H_d$$

$$\approx \frac{1}{\lg 3} \cdot n \ln d$$

$$\approx \frac{1}{\lg 3} \cdot n(\ln(\log_3 n))$$

$$= \Theta(n \ln \ln n)$$