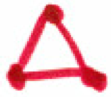
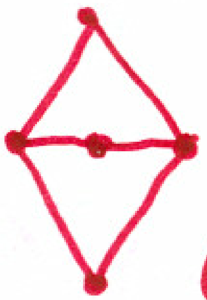
**def** A graph $G$ is <u>triangle-free</u> if it does not contain  as a subgraph.

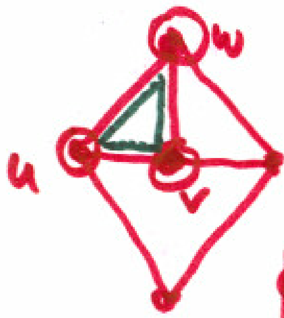
(In other words, we cannot find a set $S \subseteq V(G)$ with $|S| = 3$ and every pair of vertices in $S$ adjacent.)

<u>Ex</u>:



$G$

$G$ is triangle-free




$H$

$H$ is not triangle-free

$S = \{u, v, w\}$ forms a triangle.

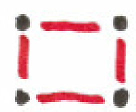
**Question:** How many edges can an $n$-vertex graph $G$ have and still not have a triangle ($\Delta$) as a subgraph?

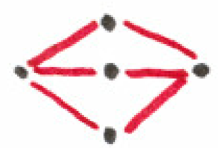
**Ex:**

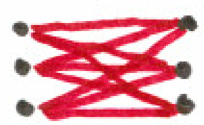
$n=1$  $0$ edges

$n=2$  $1$ edge

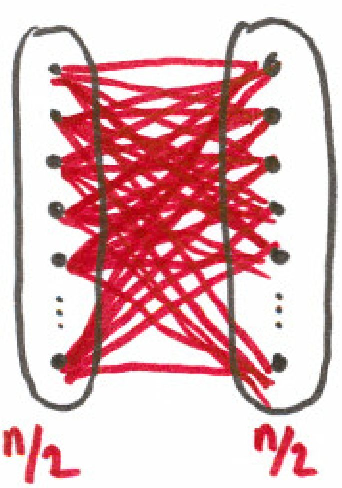
$n=3$  $2$ edges

$n=4$  $4$ edges

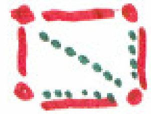
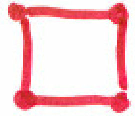
$n=5$  $6$ edges

$n=6$  $9$ edges

⋮

$n$ 

$\left\lfloor \frac{n^2}{4} \right\rfloor$ edges

$n=4$   4 edges

- <u>4 is best possible:</u> Let $G$ be a triangle-free graph on 4 vertices.

<u>CASE 1:</u> $G$ has a vertex of degree 3


$\Rightarrow G$ has $\leq 3$ edges

<u>CASE 2:</u>

$\forall u \in V(G) \quad d(u) \leq 2$

$$|E(G)| = \frac{1}{2} \sum_{u \in V(G)} d(u) \leq \frac{1}{2} \cdot \sum_{u \in V(G)} 2$$

$$= \frac{1}{2} \cdot \underbrace{2 \cdot 4} = 4$$

$\Rightarrow G$ has $\leq 4$ edges

After playing around, trying other examples, we might begin to <u>conjecture</u>:

<u>Conj</u> A triangle-free graph on $n$ vertices has at most $n^2/4$ edges.

- How can we prove this?

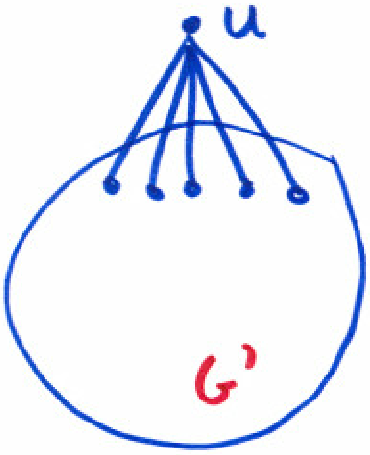
- Any ideas?

# Problem Solving Strategy:

- Find easy cases
- Pick some parameter of the problem; understand why the ~~th~~ statement is plausible at both ~~extremes~~ extreme settings of the parameter.

**Ex:** Suppose we want to ~~apply~~ use induction. The Base case will be easy.

What about the inductive step?




If $u \in V(G)$ and $G' = G - u$, then $G'$ is also triangle-free and has less vertizes than $G$, so our inductive hypothesis will tell us that $G'$ has at most $\frac{(n-1)^2}{4}$ edges. So $|E(G)| = |E(G')| + d(u) \le \frac{(n-1)^2}{4} + d(u)$.

Therefore, if $d(u)$ is small enough so that

$$\frac{(n-1)^2}{4} + d(u) \le \frac{n^2}{4}$$

we will be able to complete the inductive step for $G$.

Therefore, it is an <u>easy case</u> if $G$ contains a vertex $u$ such that

$$d(u) \le \frac{n^2}{4} - \frac{(n-1)^2}{4} = \frac{n^2 - (n^2 - 2n + 1)}{4}$$

$$= \frac{2n-1}{4}$$

So, if we focus on the <u>minimum degree</u> of $G$, we have seen the inductive step is easy when the minimum degree is small.

What if the minimum degree is large?

What if $\forall u \in V(G)\ d(u) > \frac{2n-1}{4}$ ?

First, a little trick:

If $d(u) > \frac{2n-1}{4}$, then

$$\underbrace{4 \cdot d(u)}_{\text{integer}} > \underbrace{2n-1}_{\text{integer}}$$

so in fact

$$\begin{aligned} 4 \cdot d(u) &\geq (2n-1) + 1 \\ &= 2n \end{aligned}$$

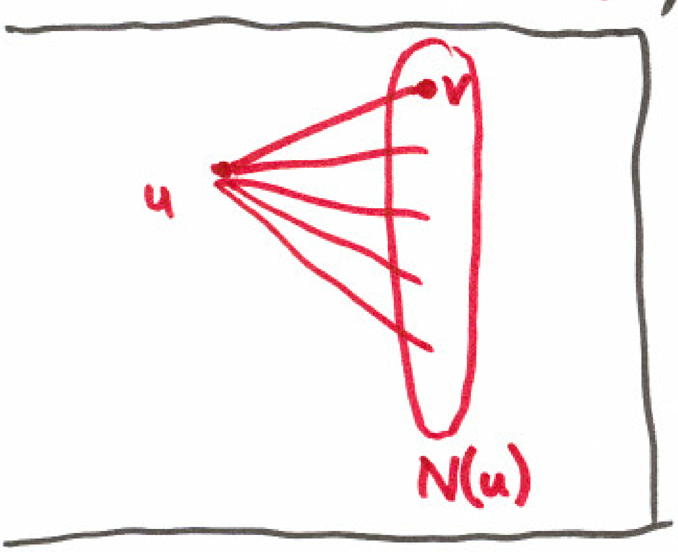
and therefore $d(u) \geq \frac{2n}{4} = \frac{n}{2}$.

So, the remaining case is: what if

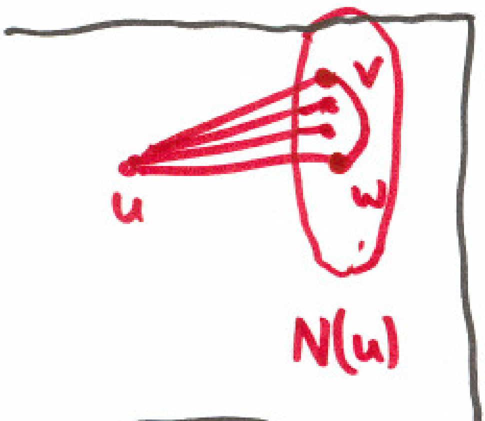
$\forall u \in V(G) \quad d(u) \geq \frac{n}{2}$ ?

Let $u \in V(G)$; remember our notation for the set $N(u) = \{ v \in V(G) \mid v \text{ is adjacent to } u \}$ of neighbors of $u$.

Because $d(u) \geq \frac{n}{2}$, we have $|N(u)| = d(u) \geq \frac{n}{2}$.




Let $v \in N(u)$. We also know $|N(v)| \geq \frac{n}{2}$. Key observation: $N(u) \cap N(v) = \emptyset$. Indeed, if $u$ and $v$ had a common neighbor $w \in N(u) \cap N(v)$, then $\{u, v, w\}$ form a triangle in $G$ which is not allowed.



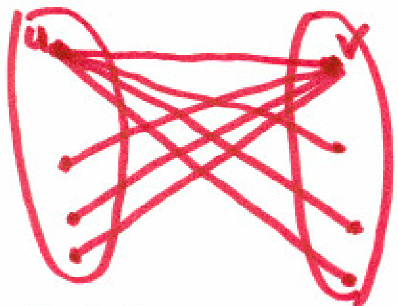

So, we have:

$$
\begin{aligned}
n = \frac{n}{2} + \frac{n}{2} &\leq |N(u)| + |N(v)| \\
&= |N(u) \cup N(v)| \\
&\leq n
\end{aligned}
$$

Because $N(u) \cap N(v) = \emptyset$

Therefore $|N(u)| = |N(v)| = \frac{n}{2}$ and $V(G)$ is the disjoint union

$$V(G) = N(u) \cup N(v).$$




Because each edge has one endpoint in $N(u)$ and one endpoint in $N(v)$, (why?), $|E(G)| \le |N(u) \times N(v)| = \frac{n^2}{4}$,

so we are done.

<u>Thm</u> If $G$ is a triangle-free graph with $n$ vertices, then $|E(G)| \le \frac{n^2}{4}$.

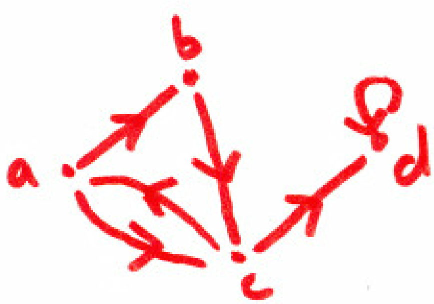
<u>Pf</u>: Exercise. Perhaps Exam 2?

**def** A <u>directed graph</u> or <u>digraph</u> $D$ is a pair of two sets:

- $V(D)$ – the vertices of $D$
- $E(D)$ – the edges of $D$

such that each $e \in E(D)$ is an <u>ordered pair</u> of [redacted] vertices.

<u>Ex:</u>




- $V(D) = \{a, b, c, d\}$
- $E(D) = \{(a,b), (b,c), (a,c), (c,a), (c,d), (d,d)\}$

<u>Think:</u> Digraphs are graphs whose edges have an orientation or direction.

<u>Remark:</u> Most graph concepts have directed graph analogs.

**def** Special Digraphs

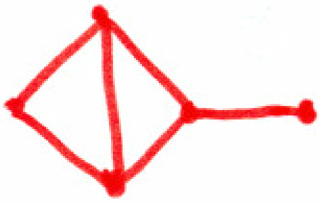
- Directed paths, $\vec{P}_n$


- Directed Cycles, $\vec{C}_n$

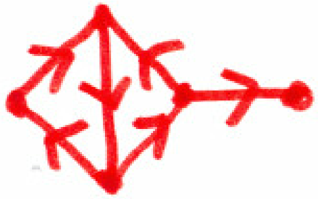

**def** An orientation of a graph $G$ is a digraph $D$ such that

- $V(D) = V(G)$
- $(u,v) \in E(D) \Rightarrow \{u,v\} \in E(G)$
- $\{u,v\} \in E(G) \Rightarrow (u,v) \in E(D)$ or $(v,u) \in E(D)$ but not both.

Ex:



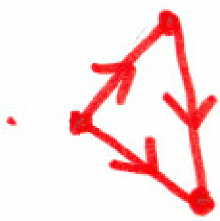
G



is an orientation of G

**def** A **tournament** is an orientation of a clique.

Ex:




1 2 3 4 is a Hamiltonian path

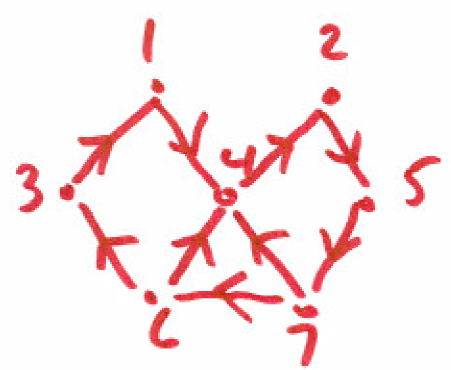
**def** ~~A path/ in~~

A (directed) path/cycle in a (directed) graph G is said to be **Hamiltonian** if the path/cycle uses all vertizes in G.

A ~~graph~~ (directed) graph G is said to be **Hamiltonian** if G contains a Hamiltonian cycle.

Compare this with the condition "G has an Eulerian ~~ex~~ trail/circuit."

Ex:




- G has a Hamiltonian cycle ~~path~~: 31425763