CSTBC Homework 3

June 15, 2007

1 People Parity

Let

$$S = \{p \,|\, p \text{ is a person who has shaken hands with an odd number of people}\}.$$

Prove that $|S|$ is even.

Solution Define a graph G by setting $V(G) = \{p \,|\, p \text{ is a person}\}$ and

$$E(G) = \{\{p_1, p_2\} \,|\, p_1 \text{ and } p_2 \text{ have shaken hands}\}.$$

Note that $S \subseteq V(G)$ is the set of all vertices in G that have odd degree. From Lecture 3, we know that $\sum_{v \in V(G)} d(v) = 2\,|E|$ and therefore $\sum_{v \in V(G)} d(v)$ is even. Suppose for a contradiction that $|S|$ is odd. Then $\sum_{v \in V(G)} d(v)$ is the sum of some even numbers and an odd number of odd numbers, which implies that $\sum_{v \in V(G)} d(v)$ is odd. Because we have reached a contradiction, it must be that $|S|$ is even.

2 Party of Five

Show that it is possible for a party of five people to gather in such a way that no three people are mutual friends or mutual strangers. That is, describe a graph G on 5 vertices with the property that for each $S \subseteq V(G)$ with $|S| = 3$, S is neither pairwise adjacent nor an independent set.

Solution Define a graph G by $V(G) = \{1, 2, 3, 4, 5\}$ and $E(G) = \{12, 23, 34, 45, 51\}$. This graph is called the 5-cycle and is denoted C_5.

3 Points in the Plane

Let P be a set of $n \geq 2$ points in the plane. Prove that for each $t \geq 0$, there exist two points p and q in P with the property that the number of points in P within distance t of p is equal to the number of points in P within distance t of q.

Solution Let $t \geq 0$ be a number. Define a graph G by $V(G) = P$ and

$$E(G) = \{\{p, q\} \,|\, p \text{ and } q \text{ are at distance at most } t\}.$$

From Lecture 3, we know that there are two vertices p and q in G with the same degree. But the degree of p is exactly the number of points in P that are within distance t of p. It follows that p and q are the desired points.

4 Sum-free Sets

A set S of numbers contains a sum if there exist $a, b, c \in S$ such that $a + b = c$ (note that a, b, and c are not necessarily distinct), and S is sum-free if it does not contain a sum. Let A be a set of $n \geq 1$ positive numbers, and let k be the largest integer such that $n > 3\binom{k}{2} + 3k$. Show that there is a sum-free set $S \subseteq A$ with $|S| \geq k + 1$.

Hint: prove the statement by contradiction. Let $S \subseteq A$ be a sum-free set of maximum size and suppose that $|S| \leq k$. Note that each number in $A - S$ introduces a sum when added to S. Recalling that $|S| \leq k$, how many nonnegative numbers can there be that introduce a sum when added to S? This gives an upper bound on $|A - S|$.

Solution Let $S \subseteq A$ be a sum-free set of maximum size and suppose for a contradiction that $|S| \leq k$. Consider $z \in A - S$. Because S is a sum-free set of maximum size, z introduces a sum when added to S; let us consider the number of ways this can happen. Note that z cannot appear three times in the sum, as $z + z = z$ is impossible because z is positive. Second, if z appears twice in the sum, then $a + z = z$ is impossible (because a is positive), so it must be that $z + z = c$ for some $c \in S$, so that $z = c/2$. Finally, if z appears once in the sum, it may be that $a + b = z$ for some $a, b \in S$ or $a + z = c$ for some $a, c \in S$ with $a \neq c$. It follows that

$$z \in \{c/2 \,|\, c \in S\} \cup \{a + b \,|\, a, b \in S\} \cup \{c - a \,|\, c, a \in S \text{ and } c \neq a\}$$

and therefore

$$A - S \subseteq \{c/2 \,|\, c \in S\} \cup \{a + b \,|\, a, b \in S\} \cup \{c - a \,|\, c, a \in S \text{ and } c \neq a\}.$$

It follows that

$$|A - S| \leq |\{c/2 \,|\, c \in S\}| + |\{a + b \,|\, a, b \in S\}| + |\{c - a \,|\, c, a \in S \text{ and } c \neq a\}|.$$

Note $|\{c/2 \,|\, c \in S\}| = |S| \leq k$. Similarly,

$$\{a + b \,|\, a, b \in S\} = \{a + a \,|\, a \in S\} \cup \{a + b \,|\, a, b \in S \text{ and } a \neq b\},$$

so that $|\{a + b \,|\, a, b \in S\}| \leq k + \binom{k}{2}$. Finally, $|\{c - a \,|\, c, a \in S \text{ and } c \neq a\}| \leq k(k-1) = 2\binom{k}{2}$. It follows that $|A - S| \leq 3\binom{k}{2} + 2k$ and therefore $|A| \leq 3\binom{k}{2} + 3k$, which contradicts $|A| > 3\binom{k}{2} + 3k$.

5 Walks and Paths

Let G be a graph and let u and v be two vertices in G. Prove that if G contains a uv-walk, then G contains a uv-path.

Solution Let W be a uv-walk in G of minimum length. We claim that W is in fact a path. Write $W = u_1 u_2 \cdots u_k$ with $u_1 = u$ and $u_k = v$; note that the length of W is $k - 1$. Suppose for a contradiction that W is not a path. It follows that $u_i = u_j$ for some $i < j$. But now removing the redundant part of W that travels from u_i back to itself yields a uv-walk

$$W' = u_1 u_2 \cdots u_{i-1} u_i u_{j+1} \cdots u_k$$

of length $k - (j - i) < k$, which contradicts that W is a uv-walk in G of minimum length. Therefore W is a uv-path in G.