CSTBC Homework 3

June 15, 2007

1 People Parity

Let

$$S = \{p \mid p \text{ is a person who has shaken hands with an odd number of people}\}.$$

Prove that $|S|$ is even.

2 Party of Five

Show that it is possible for a party of five people to gather in such a way that no three people are mutual friends or mutual strangers. That is, describe a graph G on 5 vertices with the property that for each $S \subseteq V(G)$ with $|S| = 3$, S is neither pairwise adjacent nor an independent set.

3 Points in the Plane

Let P be a set of $n \geq 2$ points in the plane. Prove that for each $t \geq 0$, there exist two points p and q in P with the property that the number of points in P within distance t of p is equal to the number of points in P within distance t of q.

4 Sum-free Sets

A set S of numbers contains a sum if there exist $a, b, c \in S$ such that $a + b = c$ (note that a, b, and c are not necessarily distinct), and S is sum-free if it does not contain a sum. Let A be a set of $n \geq 1$ positive numbers, and let k be the largest integer such that $n > 3\binom{k}{2} + 3k$. Show that there is a sum-free set $S \subseteq A$ with $|S| \geq k + 1$.

Hint: prove the statement by contradiction. Let $S \subseteq A$ be a sum-free set of maximum size and suppose that $|S| \leq k$. Note that each number in $A - S$ introduces a sum when added to S. Recalling that $|S| \leq k$, how many nonnegative numbers can there be that introduce a sum when added to S? This gives an upper bound on $|A - S|$.

5 Walks and Paths

Let G be a graph and let u and v be two vertices in G. Prove that if G contains a uv-walk, then G contains a uv-path.