CSTBC Homework 2

8th June 2007

1 A Bijection

Let $n \geq 1$ be an integer, let $\mathcal{U} = [n]$ (recall that $[n] = \{1, 2, \ldots, n\}$), and define

$$\begin{aligned} A &= [n] \times [n-1] \times \cdots \times [1] \\ B &= \{\pi \mid \pi \text{ is a permutation of } \mathcal{U}\}. \end{aligned}$$

Construct a bijection $f : A \to B$. (See hints in Lecture 2.)

Solution Let $(a_1, a_2, \ldots, a_n) \in [n] \times [n-1] \times \cdots \times [1]$. We construct a permutation $\pi = f((a_1, a_2, \ldots, a_n))$ with the following algorithm, which has n stages. Initially, let $S = [n]$; throughout the algorithm, we use S to represent the set $\{j \in [n] \mid \pi(j) \text{ is not yet defined}\}$. (In our informal description of the bijection in Lecture 2, the set of blank spaces corresponds with S.) For $1 \leq k \leq n$, the kth stage proceeds as follows. Let r be the a_kth smallest element of S, and define $\pi(r) = k$. Update $S \leftarrow S - \{r\}$, $k \leftarrow k + 1$, and proceed to the next stage. Note that because π takes on all values between 1 and n, we have that $\pi : \mathcal{U} \to \mathcal{U}$ is surjective; because $\mathcal{U}$ is finite, π must also be injective and hence a permutation. Therefore $\pi \in B$ and so f is a function $f : A \to B$.

It remains to show that f is a bijection; first, we show that f is injective. Let $(a_1, a_2, \ldots, a_n) \in A$, $(b_1, b_2, \ldots, b_n) \in A$ with $(a_1, a_2, \ldots, a_n) \neq (b_1, b_2, \ldots, b_n)$. Let k be the least integer so that $a_k \neq b_k$. Note that for each $j < k$, $a_j = b_j$, and therefore the first $k - 1$ stages of the algorithm are identical in the computation of $\pi_a = f(a_1, \ldots, a_n)$ and $\pi_b = f(b_1, \ldots, b_n)$. Consider the set S at the beginning of the kth stage in the computation of π_a and π_b. Let r_a be the a_kth smallest element of A and let r_b be the b_kth smallest element of A. Because $a_k \neq b_k$, $r_a \neq r_b$. During the kth stage of their respective computations, our algorithm sets $\pi_a(r_a) = k$ and $\pi_b(r_b) = k$. Because $r_a \neq r_b$ but $\pi_a(r_a) = k = \pi_b(r_b)$, we have $\pi_a \neq \pi_b$. Therefore f is injective.

Next, we show that f is surjective. Let π be a permutation of $\mathcal{U}$. For $1 \leq k \leq n$, define a_k to be the number of elements of π which are both as large as k and do not appear after k. That is, writing $\pi = x_1 x_2 \cdots x_n$ and $x_s = k$, we set $a_k = |\{t \in [n] \mid x_t \geq k \text{ and } t \leq s\}|$. Because $f(a_1, \ldots, a_n) = \pi$, we have that f is surjective.

2 Binomial Coefficients

By using bijections or counting the size of a set in two different ways, prove the following equalities.

1. $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.
2. $k\binom{n}{k} = n\binom{n-1}{k-1}$.
3. $\sum_{j=1}^{n} j(j-1) = 2\binom{n+1}{3}$.

Solution

1. Let $\mathcal{A} = \mathcal{P}([n])$, and for $0 \leq k \leq n$, let $\mathcal{A}_k = \{A \subseteq [n] \mid |A| = k\}$. Because $\mathcal{A}$ is the disjoint union $\mathcal{A} = \bigcup_{k=0}^{n} \mathcal{A}_k$, we have that $2^n = |\mathcal{A}| = \sum_{k=0}^{n} |\mathcal{A}_k| = \sum_{k=0}^{n} \binom{n}{k}$.

2. Let S be the set of ways to choose k elements from a set of size n together with a distinguished element. That is, $S = \{(a, A) \mid |A| \subseteq [n],\ |A| = k, \text{ and } a \in A\}$. We show that $|S| = k\binom{n}{k}$ and $|S| = n\binom{n-1}{k-1}$. For each $A \subseteq [n]$ with $|A| = k$, we have k choices for $a \in A$ so that $(a, A) \in S$. Therefore $|S| = k\binom{n}{k}$. Also, for each $a \in [n]$, there are $\binom{n-1}{k-1}$ ways to choose a set $A_0 \subseteq [n] - \{a\}$ so that $(a, A_0 \cup \{a\}) \in S$. Therefore $|S| = n\binom{n-1}{k-1}$.

3. Let $S = [2] \times \{A \subseteq [n+1] \mid |A| = 3\}$. For $1 \leq j \leq n$, let

$$S_j = \{(b, A) \in S \mid \text{the largest element of } A \text{ is } j+1\}.$$

Note that $S_j = [2] \times \{A \subseteq [j] \mid |A| = 2\}$ and so $|S_j| = 2\binom{j}{2} = j(j-1)$. Also, S is the disjoint union $S = \bigcup_{j=1}^{n} S_j$. It follows that

$$2\binom{n+1}{3} = |S| = \sum_{j=1}^{n} |S_j| = \sum_{j=1}^{n} j(j-1).$$