CSTBC Homework 1

19th June 2007

1 How Many?

Let $A = \{n \mid 1 \leq n \leq 2007 \text{ and } n \text{ is divisible by 2 or 5}\}$. Compute $|A|$.

(Hint: let

$$B = \{n \mid 1 \leq n \leq 2007 \text{ and } n \text{ is divisible by 2}\}$$

and

$$C = \{n \mid 1 \leq n \leq 2007 \text{ and } n \text{ is divisible by 5}\}.$$

What are $|B|$, $|C|$, and $|B \cap C|$?)

2 Injective? Surjective?

For each function below, determine whether the function $f : A \to B$ is bijective, injective but not surjective, surjective but not injective, or neither injective nor surjective. In problems 6-8, $n \geq 1$ is an integer.

1. $A = \{0, 1, 2, \ldots\}$, $B = \{0, -1, -2, \ldots\}$, $f(n) = -n$.
2. $A = \{0, 1, 2, \ldots\}$, $B = \{0, 1, 2, \ldots\}$, $f(n) = n + 1$.
3. $A = \{0, 1, 2, \ldots\}$, $B = \{1, 2, 3, \ldots\}$, $f(n) = n + 1$.
4. $A = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, $B = \{0, 1, 2, \ldots\}$, $f(n) = n^2$.
5. $A = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, $B = \{0, 1, 2, \ldots\}$, $f(n) = |n|$. (Recall that for a real number x, we denote the absolute value of x by $|x|$. That is, if $x \geq 0$, then $|x| = x$ and $|x| = -x$ otherwise.)
6. $\mathcal{U} = \{1, 2, \ldots, n\}$, $A = B = \mathcal{P}(\mathcal{U})$, $f(S) = \overline{S}$.
7. $\mathcal{U} = \{1, 2, \ldots, n\}$, $A = B = \mathcal{P}(\mathcal{U})$, $f(S) = S \cup \{1\}$.
8. $\mathcal{U} = \{1, 2, \ldots, n\}$, $A = B = \mathcal{P}(\mathcal{U})$,

$$f(S) = \begin{cases} S \cup \{1\} & 1 \notin S \\ S - \{1\} & 1 \in S \end{cases}.$$

3 An Injection

Let $n \geq 1$ be an integer, let $\mathcal{U} = \{1, 2, \ldots, n\}$, and let $\mathcal{A} = \{A \subseteq \mathcal{U} \mid |A| = k\}$; that is, $\mathcal{A}$ consists of all the sets $A \subseteq \mathcal{U}$ which have size k. Construct an injection $f : \mathcal{A} \to \mathcal{U}^k$. What can we conclude about $|\mathcal{A}|$?

4 Pairwise Disjoint Families

Let $n \geq 1$ be an integer and let $\mathcal{U} = \{1, 2, \ldots, n\}$. We say that a family $\mathcal{A} \subseteq \mathcal{P}(\mathcal{U})$ of sets is pairwise disjoint if, for each pair of sets A, $B \in \mathcal{A}$, we have that A and B are disjoint (that is, $A \cap B = \emptyset$).

1. Prove that if $\mathcal{A} \subseteq \mathcal{P}(\mathcal{U})$ is a pairwise disjoint family of sets, then $|\mathcal{A}| \leq n + 1$.
2. Find a pairwise disjoint family $\mathcal{A} \subseteq \mathcal{P}(\mathcal{U})$ with $|\mathcal{A}| = n + 1$.
3. Besides the family $\mathcal{A}$ that you found in part (2), are there any other pairwise disjoint families $\mathcal{B} \subseteq \mathcal{P}(\mathcal{U})$ with $|\mathcal{B}| = n + 1$?

5 More Pairwise Intersecting Families

Let $n \geq 1$ be an integer and let $\mathcal{U} = \{1, 2, \ldots, n\}$. Recall from lecture 1 that a family $\mathcal{A} \subseteq \mathcal{P}(\mathcal{U})$ of sets is pairwise intersecting if, for each pair of sets A, $B \in \mathcal{A}$, we have that $A \cap B \neq \emptyset$. In lecture 1, we saw that if $\mathcal{A} \subseteq \mathcal{P}(\mathcal{U})$ is pairwise intersecting, then $|\mathcal{A}| \leq 2^{n-1}$. We also found that $\mathcal{A} = \{A \subseteq U \mid 1 \in A\}$ is an example of a pairwise intersecting family of size 2^{n-1}, but this family has the property that there exists an element $j \in \mathcal{U}$ (namely, $j = 1$), such that for each $A \in \mathcal{A}$, $j \in A$.

Construct a pairwise intersecting family $\mathcal{B} \subseteq \mathcal{P}(\mathcal{U})$ of size $|\mathcal{B}| = 2^{n-1}$ which fails to have this property. That is, you are asked to find a family $\mathcal{B} \subseteq \mathcal{P}(\mathcal{U})$ with the following properties:

1. $\mathcal{B}$ is pairwise intersecting,
2. $|\mathcal{B}| = 2^{n-1}$, and
3. for each $j \in \mathcal{U}$, there exists some $B \in \mathcal{B}$ such that $j \notin B$.

(Hint: you may find the proof that $|\mathcal{B}| \leq 2^{n-1}$ helpful.)