CSTBC Exam 1 Solutions
Due: June 25, 2007

June 26, 2007

This exam is open notes/open lecture and covers material from lectures 1-6. You are welcome to use any of the course material linked from the CSTBC website. You should not use other reference materials. If you have any questions, please ask me.

1 How Many?

Let $A = \{1, 2, \ldots, m\}$ and $B = \{1, 2, \ldots, n\}$.

1. What is $|A \cup B|$?
2. What is $|A \cap B|$?
3. What is $|A - B|$? (Warning: consider carefully the cases $m \geq n$ and $m < n$.)
4. What is $|\{f \mid f : A \to B \text{ is a function from } A \text{ to } B\}|$? (Hint: first try to solve the problem for some small values of m and n. For example, how many functions are there if $m = n = 1$? What about $m = 3$ and $n = 2$? Try some more examples. Do you see a general pattern? Does your guess work for all the examples you have tried? Can you prove that your guess is correct?)
5. What is $|\{G \mid G \text{ is a graph with } V(G) = A\}|$? (Hint: the same strategies as in part (4) apply. Try to solve the problem with $m = 1$, $m = 2$, and $m = 3$.)

Solution

1. $|A \cup B| = \max\{m, n\}$.
2. $|A \cap B| = \min\{m, n\}$.
3. $|A - B| = \max\{0, m - n\}$.
4. For each element $a \in A$, we have n choices for $f(a) \in B$. Hence there is a natural bijection from the set of all functions $f : A \to B$ to the set $\{(b_1, b_2, \ldots, b_m) \mid \text{each } b_j \in B\} = [n]^m$. Therefore the number of functions $f : A \to B$ is n^m.
5. A graph G consists of a set of vertices $V(G)$ and a set of edges $E(G)$. Let $X = \{\{u, v\} \mid u, v \in V(G)\}$ be the set of (unordered) pairs of vertices of G. By definition, $E(G)$ is a subset of X. Therefore there are $|\mathcal{P}(X)| = 2^{|X|} = 2^{\binom{m}{2}}$ possible ways to choose $E(G)$ and hence $2^{\binom{m}{2}}$ graphs with vertex set A.

2 Practice Problem from Lecture 2

In lecture 2, we prove that if $0 \leq k \leq n$, $\binom{n}{k} = \binom{n}{n-k}$. Do the practice problem associated with this theorem; that is, for $n = 5$ and $k = 2$, explicitly write down $\mathcal{A} = \{A \subseteq \mathcal{U} \mid |A| = k\}$, $\mathcal{B} = \{B \subseteq \mathcal{U} \mid |B| = n - k\}$, and the bijection $f : \mathcal{A} \to \mathcal{B}$.

Solution We describe $\mathcal{A}$, $\mathcal{B}$, and f in the following table. The left column lists the elements of $\mathcal{A}$, the right column lists the elements of $\mathcal{B}$.

A	$f(A)$
$\{1,2\}$	$\{3,4,5\}$
$\{1,3\}$	$\{2,4,5\}$
$\{1,4\}$	$\{2,3,5\}$
$\{1,5\}$	$\{2,3,4\}$
$\{2,3\}$	$\{1,4,5\}$
$\{2,4\}$	$\{1,3,5\}$
$\{2,5\}$	$\{1,3,4\}$
$\{3,4\}$	$\{1,2,5\}$
$\{3,5\}$	$\{1,2,4\}$
$\{4,5\}$	$\{1,2,3\}$

3 Injective, Surjective, Bijective

Let A, B, C be sets and let $f : A \to B$, $g : B \to C$ be functions. Define the function $h : A \to C$ by setting $h(a) = g(f(a))$ for all $a \in A$; in words, the function h maps $a \in A$ to an element in C by first applying f to a to obtain an element $b = f(a)$ in B, and then applying g to b to obtain an element $c = g(b)$ in C. We call h the composition of f and g, and we write $h = g \circ f$.

Decide whether each of the following statements are necessarily true, or not necessarily true (false).

1. If g is an injection, then h is an injection.
2. If g is a surjection, then h is a surjection.
3. If f is an injection and g is a surjection, then h is a bijection.
4. If f is an injection and g is an injection, then h is an injection.
5. If h is a bijection, then f and g are bijections.
6. If h is an injection, then f and g are injections.
7. If h is a surjection, then f and g are surjections.

Solution

1. false
2. false. Here's a counterexample: $A = \{1\}$, $B = C = \{1, 2, 3\}$, $f(a) = a$, and $g(b) = b$.
3. false
4. true
5. false

6. false; this one is a little tricky: if h is an injection, then f must be an injection; however, g might not be an injection. For example, take $A = \{1, 2, 3\}$, $B = \{1, 2, 3, 4\}$, $C = \{1, 2, 3\}$, define $f : A \to B$ by $f(a) = a$, and define $g : B \to C$ by
$$g(b) = \begin{cases} b & b < 4 \\ 1 & b = 4 \end{cases}.$$
Notice $h : A \to C$ is the identity function and hence h is injective. However, g is not an injection, because $g(1) = g(4) = 1$.

7. false

4 An Equality

Give two different proofs of the following equality: for all $n \geq 0$, $\sum_{j=0}^{n} 2^j = 2^{n+1} - 1$.

1. Let $\mathcal{U} = \{1, 2, 3, \ldots, n, n+1\}$, let $\mathcal{A} = \{A \subseteq \mathcal{U} \mid A \neq \emptyset\}$, and for $0 \leq j \leq n$, let
$$\mathcal{A}_j = \{A \subseteq \mathcal{U} \mid \text{the largest element in } A \text{ is } j+1\}.$$
Use these sets to establish the equality.

2. Prove the equality by induction on n.

Solution

1. First, note that $\mathcal{A}$ consists of all the nonempty subsets of $\mathcal{U}$. Therefore, $\mathcal{A} = \mathcal{P}(\mathcal{U}) - \{\emptyset\}$ and so $|\mathcal{A}| = 2^{n+1} - 1$. We can also count $|\mathcal{A}|$ differently, by grouping each set $A \in \mathcal{A}$ according to the largest element in A. That is, $\mathcal{A}$ is the disjoint union $\mathcal{A} = \mathcal{A}_0 \cup \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_n$ where each set $A \in \mathcal{A}$ is a member of the group $\mathcal{A}_j$ if and only if the largest element in A is $j+1$. Thus, $\mathcal{A}_0 = \{\{1\}\}$, $\mathcal{A}_1 = \{\{2\}, \{1, 2\}\}$, $\mathcal{A}_3 = \{\{3\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$, etc. Observe that $\mathcal{A}_j$ is formed by taking the collection of all subsets of $\{1, 2, \ldots, j\}$ and adding the element $j+1$ to each one of them. Therefore $|\mathcal{A}_j| = |\mathcal{P}(\{1, 2, \ldots, j\})| = 2^j$. Therefore, we have
$$\begin{aligned} 2^{n+1} - 1 = |\mathcal{A}| &= \sum_{j=0}^{n} |\mathcal{A}_j| \\ &= \sum_{j=0}^{n} 2^j \end{aligned}$$
and the statement is proved.

2. Proof: by induction on n. Base case: if $n = 0$, then $2^{0+1} - 1 = 1 = \sum_{j=0}^{0} 2^j$, so the result holds. Inductive step: let $n \geq 1$. We compute
$$\sum_{j=0}^{n} 2^j = \left(\sum_{j=0}^{n-1} 2^j \right) + 2^n.$$
By the inductive hypothesis, $\sum_{j=0}^{n-1} 2^j = 2^{(n-1)+1} - 1 = 2^n - 1$. Therefore
$$\begin{aligned} \sum_{j=0}^{n} 2^j &= \left(\sum_{j=0}^{n-1} 2^j \right) + 2^n \\ &= 2^n - 1 + 2^n \\ &= 2^{n+1} - 1 \end{aligned}$$
and the proof is complete.

5 Graphs

A graph G is connected if for each pair of vertices u and v in G, there is a uv-walk in G. Prove that G is connected if and only if for each $S \subseteq V(G)$ with $S \neq \emptyset$ and $S \neq V(G)$, there is an edge in G with one endpoint inside S and one endpoint outside S.

Notes This question asks us to prove an "if and only if" statement. That means we must prove two mathematical statements. First, we must prove that if G is connected, then for each $S \subseteq V(G)$ with $S \neq \emptyset$ and $S \neq V(G)$, there is an edge in G with one endpoint inside S and one endpoint outside S. Because of the order that the statement is written, this is called the forward direction and is marked ($\implies$) in our solution below. Second, we must prove that if G has the property that for each $S \subseteq V(G)$ with $S \neq \emptyset$ and $S \neq V(G)$, there is an edge in G with one endpoint inside S and one endpoint outside S, then G is connected. This is called the backward direction, or converse, and is marked ($\impliedby$) in our solution below.

Solution ($\implies$). Let G be a connected graph and suppose for a contradiction that there exists $S \subseteq V(G)$ with $S \neq \emptyset$ and $S \neq V(G)$ so that there are no edges with one endpoint inside S and one endpoint outside S. Because $S \neq \emptyset$, there is a vertex $u \in S$. Because $S \neq V(G)$, there is a vertex $v \notin S$. Because G is connected, there is a uv-walk $W = w_0 w_1 \cdots w_k$ in G (note that $w_0 = u$ and $w_k = v$). Because W starts at a vertex $w_0 = u$ inside S and ends at a vertex $w_k = v$ outside S, there must be some $0 \leq j \leq k-1$ such that w_j is in S but w_{j+1} is outside S. Because W is a walk, $\{w_j, w_{j+1}\}$ is an edge in G with one endpoint inside S and one endpoint outside S, and this is a contradiction.

($\impliedby$). Let G be a graph with the property that for each $S \subseteq V(G)$ with $S \neq \emptyset$ and $S \neq V(G)$, there is an edge in G with one endpoint in S and one endpoint outside S. We show that G is connected. Consider a pair of vertices u, v in G. We must show that there is a uv-walk in G. Define the set

$$S = \{w \in V(G) \mid \text{there exists a } uw\text{-walk in } G\}$$

of all vertices w in G to which it is possible to walk, starting from u. Note that the walk $W = u$ is a uu-walk in G, and so $u \in S$. We claim that $v \in S$ also. Suppose for a contradiction that $v \notin S$. Because $u \in S$, $S \neq \emptyset$. Because $v \notin S$, $S \neq V(G)$. By assumption, there is an edge $\{w, x\}$ in G with one endpoint (call it w) inside S and one endpoint (call it x) outside S. Because $w \in S$, there exists a uw-walk W in G. Because $\{w, x\} \in E(G)$, we can walk from u to x in G by first following W from u to w and then traversing the edge $\{w, x\}$ from w to x. Therefore G contains a ux-walk, and so $x \in S$. But this contradicts that x is a vertex outside S. The contradiction implies that $v \in S$ as required.

6 A Proof with an Error

The following inductive "proof" contains an error. What is the number of the first line in the proof that is incorrect? Why is it incorrect?

Theorem: If $n \geq 1$ balls are placed into a box B and each ball is colored blue or yellow, then either all balls in B are blue or all balls in B are yellow.

Proof:

1. By induction on n.
2. Base case: If $n = 1$, then $|B| = 1$, so the theorem is clearly true in this case.
3. Inductive Step: suppose $n \geq 2$.
4. Let $x, y \in B$ be two distinct balls in B.
5. Let $B_1 = B - \{x\}$ and $B_2 = B - \{y\}$. Note that $|B_1| = |B_2| = n - 1 < n$.

6. Therefore, the inductive hypothesis implies that all balls in B_1 are blue or all balls in B_1 are yellow.
7. Similarly, the inductive hypothesis implies that all balls in B_2 are blue or all balls in B_2 are yellow.
8. Note that the common color of all balls in B_1 must be the same as the common color of all balls in B_2.
9. Because $B = B_1 \cup B_2$ and the common color in B_1 is the same as the common color in B_2, all balls in B are blue or all balls in B are yellow.

Solution The first line that is in error is line (8). The error is subtle, however: line (8) implicitly assumes that B_1 and B_2 share common balls; that is, line (8) assumes $B_1 \cap B_2 \neq \emptyset$. Although this is true whenever we start with $n \geq 3$ balls that $B_1 \cap B_2 \neq \emptyset$, this fails for the case that $n = 2$. In this case, we apply our inductive step with $B = \{x, y\}$, and we set $B_1 = \{y\}$ and $B_2 = \{x\}$ in step (5). We correctly apply the inductive hypothesis in steps (6) and (7), but step (8) fails because the common color of the balls in B_1 (namely, the color of y) does not have to be the same as the common color of the balls in B_2 (namely, the color of x).

7 Pirates

Recently, a pirate ship with 200 pirates onboard has captured 1000 gold coins from another vessel. The pirates have developed an interesting way to distribute their gains among themselves. Here's what happens: the strongest pirate on the ship proposes a distribution of the coins to pirates. Next, all pirates vote on the proposal (including the strongest pirate). If at least half of the pirates vote in support of the proposal, the coins are distributed according to the proposal and the process is complete. However, if more than half of the pirates vote to reject the proposal, they throw the strongest pirate overboard and the process repeats with the strongest of the remaining pirates offering a new proposal. Each pirate is perfectly logical and wishes to maximize the number of coins that he or she receives.

What does the strongest pirate propose? Prove your answer is correct. (Hint: first, try to answer the question if there are only a small number of pirates onboard – one pirate, two pirates, etc. Next, based upon your investigation of what happens with a small number of pirates, guess what is proposed if there are n $(1 \leq n \leq 200)$ pirates onboard. Finally, prove by induction on n that your guess is correct.)

Notes This solution uses floor and ceiling notation. The floor of a real number x, written $\lfloor x \rfloor$, is the largest integer k such that $k \leq x$. Similarly, the ceiling of a real number x, written $\lceil x \rceil$, is the smallest integer k such that $k \geq x$.

Solution Suppose there are n pirates $p_1, p_2, \ldots, p_n$ onboard, ordered from strongest pirate p_1 to weakest pirate p_n. We claim that p_1 proposes the following distribution: $p_2, p_4, p_6, \ldots$ each receive 0 gold coins, $p_3, p_5, p_7, \ldots$ each receive 1 gold coin, and p_1 gets the remaining $1000 - \lfloor \frac{n-1}{2} \rfloor$ gold coins.

We prove the claim by induction on n. Base case: if $n = 1$, then p_1 proposes to receive all gold coins, which is consistent with our claim. Inductive step: let $n \geq 2$. Because p_1 is perfectly logical and wishes to maximize the number of gold coins p_1 receives, p_1 will reason as follows: "I need at least $n/2$ of my fellow pirates to vote in favor of my proposed distribution. If my proposal is rejected and I am thrown off the ship, p_2 will offer a new distribution. By the inductive hypothesis, I know that p_2 will offer zero gold coins to $p_3, p_5, p_7, \ldots$, one gold coin to $p_4, p_6, p_8, \ldots$, and keep the rest. Furthermore, all my fellow pirates are perfectly logical and know full well how the coins will be distributed if I'm thrown overboard. I can only be sure that a pirate p_j will vote for my proposal if I offer p_j more than p_j would get if p_2 were running the show. Because I want to get as many coins as possible, I should start by offering gold coins to the pirates that receive the fewest number of gold coins under p_2's distribution, and I should continue buying off votes until I am sure that at least $n/2$ pirates will vote for me. So, I will offer one coin to each of $p_3, p_5, p_7, \ldots$ and keep the rest. That way, I will get votes from $p_1, p_3, p_5, \ldots$, so I will get $\lceil \frac{n}{2} \rceil \geq n/2$ votes and my proposal will be accepted. Unfortunately, I can't do any better because I need at least $\lceil \frac{n}{2} \rceil$ votes and I've bought each for the cheapest possible price." Therefore p_1 makes the claimed proposal, and our proof is complete.

In our case with $n = 200$, the strongest pirate p_1 offers zero gold coins to $p_2, p_4, \ldots, p_{200}$, one gold coin to $p_3, p_5, p_7, \ldots, p_{199}$ and keeps the remaining 901 coins.

8 Ramsey Theory

In lecture 6, we present a classic proof from Ramsey theory and define $R(m, n)$ for each $m, n \geq 1$. Describe how the proof can be modified to show that $R(m, n) \leq \binom{m+n}{m}$. (Note: you are not asked to repeat the proof in full, just describe how to modify the proof we saw in lecture 6.)

Solution We modify the proof as follows.

1. Modify the statement of the theorem so that it reads "Theorem: $\forall m, n \geq 1 \, R(m, n) \leq \binom{m+n}{m}$."
2. At the end of the base case, add "Therefore $R(m, n) \leq 1$. Because $1 \leq \binom{m+n}{m}$ for $m, n \geq 1$, the theorem holds."
3. Change "by the inductive hypothesis, there exists r_1 such that ..." to "by the inductive hypothesis, there exists $r_1 = R(m-1, n) \leq \binom{m-1+n}{m-1}$ such that ...". Similarly, change "by the inductive hypothesis, there exists r_1 such that ..." to "by the inductive hypothesis, there exists $r_2 = R(m, n-1) \leq \binom{m+n-1}{m}$ such that ..."
4. At the end of the proof, append "Therefore

$$R(m, n) \leq r = r_1 + r_2 \leq \binom{m+n-1}{m-1} + \binom{m+n-1}{m} = \binom{m+n}{m},$$

where the last equality is an identity we know from lecture 2."