CSTBC Exam 1
Due: June 25, 2007

17th June 2007

This exam is open notes/open lecture and covers material from lectures 1-6. You are welcome to use any of the course material linked from the CSTBC website. You should not use other reference materials. If you have any questions, please ask me.

1 How Many?

Let $A = \{1, 2, \ldots, m\}$ and $B = \{1, 2, \ldots, n\}$.

1. What is $|A \cup B|$?
2. What is $|A \cap B|$?
3. What is $|A - B|$? (Warning: consider carefully the cases $m \geq n$ and $m < n$.)
4. What is $|\{f \mid f : A \to B$ is a function from A to $B\}|$? (Hint: first try to solve the problem for some small values of m and n. For example, how many functions are there if $m = n = 1$? What about $m = 3$ and $n = 2$? Try some more examples. Do you see a general pattern? Does your guess work for all the examples you have tried? Can you prove that your guess is correct?)
5. What is $|\{G \mid G$ is a graph with $V(G) = A\}|$? (Hint: the same strategies as in part (1) apply. Try to solve the problem with $m = 1$, $m = 2$, and $m = 3$.)

2 Practice Problem from Lecture 2

In lecture 2, we prove that if $0 \leq k \leq n$, $\binom{n}{k} = \binom{n}{n-k}$. Do the practice problem associated with this theorem; that is, for $n = 5$ and $k = 2$, explicitly write down $\mathcal{A} = \{A \subseteq \mathcal{U} \mid |A| = k\}$, $\mathcal{B} = \{B \subseteq \mathcal{U} \mid |B| = n - k\}$, and the bijection $f : \mathcal{A} \to \mathcal{B}$.

3 Injective, Surjective, Bijective

Let A, B, C be sets and let $f : A \to B$, $g : B \to C$ be functions. Define the function $h : A \to C$ by setting $h(a) = g(f(a))$ for all $a \in A$; in words, the function h maps $a \in A$ to an element in C by first applying f to a to obtain an element $b = f(a)$ in B, and then applying g to b to obtain an element $c = g(b)$ in C. We call h the composition of f and g, and we write $h = g \circ f$.

Decide whether each of the following statements are necessarily true, or not necessarily true (false).

1. If g is an injection, then h is an injection.
2. If g is a surjection, then h is a surjection.

3. If f is an injection and g is a surjection, then h is a bijection.
4. If f is an injection and g is an injection, then h is an injection.
5. If h is a bijection, then f and g are bijections.
6. If h is an injection, then f and g are injections.
7. If h is a surjection, then f and g are surjections.

4 An Equality

Give two different proofs of the following equality: for all $n \geq 0$, $\sum_{j=0}^{n} 2^j = 2^{n+1} - 1$.

1. Let $\mathcal{U} = \{1, 2, 3, \ldots, n, n+1\}$, let $\mathcal{A} = \{A \subseteq \mathcal{U} \mid A \neq \emptyset\}$, and for $0 \leq j \leq n$, let

$$\mathcal{A}_j = \{A \subseteq \mathcal{U} \mid \text{the largest element in } A \text{ is } j+1\}\,.$$

   Use these sets to establish the equality.

2. Prove the equality by induction on n.

5 Graphs

A graph G is connected if for each pair of vertices u and v in G, there is a uv-walk in G. Prove that G is connected if and only if for each $S \subseteq V(G)$ with $S \neq \emptyset$ and $S \neq V(G)$, there is an edge in G with one endpoint inside S and one endpoint outside S.

6 A Proof with an Error

The following inductive "proof" contains an error. What is the number of the first line in the proof that is incorrect? Why is it incorrect?

Theorem: If $n \geq 1$ balls are placed into a box B and each ball is colored blue or yellow, then either all balls in B are blue or all balls in B are yellow.

Proof:

1. By induction on n.
2. Base case: If $n = 1$, then $|B| = 1$, so the theorem is clearly true in this case.
3. Inductive Step: suppose $n \geq 2$.
4. Let $x, y \in B$ be two distinct balls in B.
5. Let $B_1 = B - \{x\}$ and $B_2 = B - \{y\}$. Note that $|B_1| = |B_2| = n - 1 < n$.
6. Therefore, the inductive hypothesis implies that all balls in B_1 are blue or all balls in B_1 are yellow.
7. Similarly, the inductive hypothesis implies that all balls in B_2 are blue or all balls in B_2 are yellow.
8. Note that the common color of all balls in B_1 must be the same as the common color of all balls in B_2.
9. Because $B = B_1 \cup B_2$ and the common color in B_1 is the same as the common color in B_2, all balls in B are blue or all balls in B are yellow.

7 Pirates

Recently, a pirate ship with 200 pirates onboard has captured 1000 gold coins from another vessel. The pirates have developed an interesting way to distribute their gains among themselves. Here's what happens: the strongest pirate on the ship proposes a distribution of the coins to pirates. Next, all pirates vote on the proposal (including the strongest pirate). If at least half of the pirates vote in support of the proposal, the coins are distributed according to the proposal and the process is complete. However, if more than half of the pirates vote to reject the proposal, they throw the strongest pirate overboard and the process repeats with the strongest of the remaining pirates offering a new proposal. Each pirate is perfectly logical and wishes to maximize the number of coins that he or she receives.

What does the strongest pirate propose? Prove your answer is correct. (Hint: first, try to answer the question if there are only a small number of pirates onboard – one pirate, two pirates, etc. Next, based upon your investigation of what happens with a small number of pirates, guess what is proposed if there are n $(1 \leq n \leq 200)$ pirates onboard. Finally, prove by induction on n that your guess is correct.)

8 Ramsey Theory

In lecture 6, we present a classic proof from Ramsey theory and define $R(m, n)$ for each $m, n \geq 1$. Describe how the proof can be modified to show that $R(m, n) \leq \binom{m+n}{m}$. (Note: you are not asked to repeat the proof in full, just describe how to modify the proof we saw in lecture 6.)