CSTBC Exam 2 Solutions
Due: July 30, 2007, 11:59pm

August 2, 2007

This exam is open notes/open lecture and covers material from lectures 7-17. You are welcome to use any of the course material linked from the CSTBC website. You should not use other reference materials. Please send me your solutions to the exam via email. If you have any questions, please ask me.

1 Pirate's Dinner

After distributing their treasure, our n pirates from Exam 1 have worked up an appetite. The pirate ship's cafeteria offers three different, non-overlapping dinner times. As the strongest pirate, you are in charge of assigning each pirate to one of the three dinner slots. Unfortunately, not all pirates get along with each other. You have a list of k pairs of pirates that have fought each other in the past. Prove that you can assign pirates to dinners so that at most $k/3$ of the troublesome pairs eat dinner at the same time.

Solution We model this problem using graph theory. Define a graph G whose vertices are the pirates and let uv be an edge in G if and only if u and v have fought each other; note that G has n vertices and k edges. In the language of graph theory, this problem asks us to partition $V(G)$ into three sets A, B, C so that at most $k/3$ edges have both endpoints in one of the sets A, B, C.

The proof is by induction on n. If $n = 1$, then G has zero edges and any partition works. For $n \geq 2$, we choose a vertex $u \in V(G)$ arbitrarily; let $H = G - u$ be the graph obtained from G by deleting u and all incident edges. Let $k_0 = k - d(u)$ be the number of edges in H. Because $|V(H)| < n$, we may apply the inductive hypothesis to obtain a partition A_0, B_0, C_0 of $V(H)$ so that at most $k_0/3$ edges of H have both endpoints in one of the sets A_0, B_0, C_0. We extend this partition of $V(H)$ to a partition of $V(G)$ by adding u to one of the sets A_0, B_0, C_0. In particular, we add u to whichever set contains the fewest neighbors of u; let A, B, C be the resulting partition of $V(G)$. Because u has at most $d(u)/3$ neighbors in whichever one of the sets A_0, B_0, C_0 to which it was added, G contains at most $k_0/3 + d(u)/3 = k/3$ edges with both endpoints in one of the sets A, B, C.

2 Hamiltonian Paths in Tournaments

In Lecture 12, we introduce the concept of a tournament. Prove that if T is a tournament on n vertices, then T contains a Hamiltonian path.

Solution The proof is by induction on n. If $n = 1$, the statement is clearly true. For $n \geq 2$, let u be a vertex in T and let $T_0 = T - u$ be the tournament obtained from T by deleting u and all incident edges. By the inductive hypothesis, T_0 contains a Hamiltonian path $P_0 = v_1 v_2 \cdots v_{n-1}$. We consider two cases. First, suppose that all edges incident to u are directed into u, so that u is a sink in T. In this case, $v_{n-1}u$ is an edge in T and so we can extend P_0 to a Hamiltonian path $P = v_1 v_2 \cdots v_{n-1} u$ in T by appending u. Otherwise, there is at least one edge incident to u directed out of u; let k be the least integer such that uv_k is an edge in T. If $k = 1$, then we may extend P_0 to a Hamiltonian path $P = uv_1 v_2 \cdots v_{n-1}$ by prepending u. Finally, if $k > 1$, then by our selection of k, $v_{k-1}u$ must be an edge in T. In this case, we may extend P_0 to a Hamiltonian path $P = v_1 \cdots v_{k-1} u v_k \cdots v_{n-1}$ by inserting u in between v_{k-1} and v_k.

3 A Puzzle

Let $k \geq 0$ be an integer and let $n = 2^k$. Suppose you are given a square which is divided into n^2 smaller squares. The small squares are all colored blue, except that one special square is colored red. You are also given access to green game pieces which can be placed on the board. Each game piece comes in an 'L' shape and covers three of the smaller squares. Show that you can place the game pieces on the board in such a way that the pieces do not overlap and cover all the blue squares, leaving the red square uncovered. An example with $n = 8$ appears below.

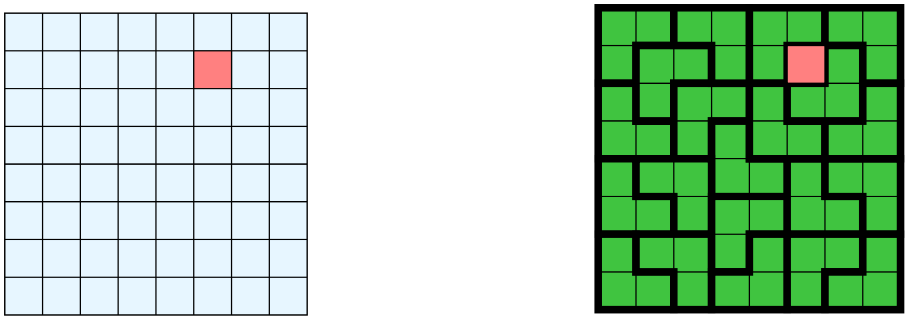

Figure 1: A puzzle and solution with $n = 8$

Solution The proof is by induction on k. If $k = 0$, the game consists of a single square, much must be colored red, and the requirements are satisfied by not placing any game pieces on the board. If $k \geq 1$, we partition the board into its 4 quadrants, each of which is a smaller $2^{k-1} \times 2^{k-1}$ board. Note that the red square is located in just one of the quadrants; the other three are filled with blue squares. By rotating the original board, we might as well assume that the red square is located in the upper-right quadrant, as in the example above.

Next, we color 3 more squares red: the lower-right corner of the upper-left quadrant, the upper-right corner of the lower-left quadrant, and the upper-left corner of the lower-right quadrant. Note that these three squares are in an 'L' shape, and each of the quadrants contains exactly one red square. By the inductive hypothesis, we may cover each of the smaller quadrants with green game pieces, leaving only the 4 red squares uncovered. Finally, we place a game piece on the 3 squares we colored red. Except for the original red square, all squares are covered with green game pieces and we are done.

4 Graphs with (almost) unique degrees

In Lecture 3, we saw that a graph G with at least two vertices must contain two vertices of the same degree. Let G be a graph that contains a vertex u such that no two vertices in $V(G) - \{u\}$ have the same degree. What can you say about the degree of u?

Notes This question appears to have been misread; I meant that no two vertices in $V(G) - \{u\}$ have the same degree in G. Another way of phrasing this question is as follows. For $n \geq 2$, let G be an n-vertex graph such that $|\{d(w) \mid w \in V(G)\}| = n-1$. By the pigeonhole principle, there are two vertices u, v in G with the property that $d(u) = d(v)$. What can you say about the degree of u?

Solution If n is odd, then $d(u) = \frac{n-1}{2}$. If n is even, then $d(u) \in \left\{\frac{n}{2} - 1, \frac{n}{2}\right\}$. The proof is by induction on n. If $n \leq 2$, the statement is clearly true. Suppose $n \geq 3$ and let $S = \{d(w) \mid w \in V(G) - \{u\}\}$. Because $S \subseteq \{0, 1, \cdots, n-1\}$, $|S| = n-1$, and it is impossible to have $0, n-1 \in S$, it must be that $S = \{0, 1, \cdots, n-2\}$ or $S = \{1, 2, \cdots, n-1\}$. First, suppose that $S = \{0, 1, \cdots, n-2\}$. Choose $w_1, w_2 \in V(G) - \{u\}$ so that $d(w_1) = 0$ and $d(w_2) = n-2$. Because w_1 is not adjacent to any vertex in G and w_2 is adjacent to all but two of the vertices in G, it must be that w_2 is adjacent to each vertex in G except for w_1 and itself. Let $G_0 = G - w_1 - w_2$ be the graph obtained from G by deleting w_1, w_2, and all incident edges. Because w_1 is not adjacent to any vertex in G_0 and w_2 is adjacent to every vertex in G_0, the degree of each vertex in G_0 is one less than its degree in G. Therefore G_0 is a graph with the property that no two vertices in $V(G_0) - \{u\}$ have the same degree in G_0. It follows from the inductive hypothesis that the degree of u in G_0 is $\frac{|V(G_0)|-1}{2} = \frac{n-3}{2}$ if n is odd and one of $\left\{\frac{n-2}{2} - 1, \frac{n-2}{2}\right\}$ of n is even. Therefore in G, the degree of u in G is $\frac{n-3}{2} + 1 = \frac{n-1}{2}$ if n is odd and one of $\left\{\frac{n-2}{2}, \frac{n}{2}\right\}$ if n is even.

The case that $S = \{1, 2, \cdots, n-1\}$ is similar: choose $w_1, w_2 \in V(G) - \{u\}$ so that $d(w_1) = 1$ and $d(w_2) = n-1$. This time, w_2 is adjacent to all other vertices in G, including w_1. Because w_1 has degree one, w_1 is adjacent only to w_2. Therefore, setting $G_0 = G - w_1 - w_2$, we again have the property that the degree of each vertex in G_0 is one less than its degree in G. The rest of the proof is identical to the first case.

5 Recurrences

5.1 An Exact Solution

In this problem, we will solve a linear homogeneous recurrence (see Lecture 16). Let

$$T(n) = \begin{matrix} 0 & n \in \{0, 1\} \\ 3T(n-1) + 10T(n-2) + n - 2 & n \geq 2 \end{matrix} .$$

1. Find the general solution to the homogeneous recurrence $S(n) = 3S(n-1) + 10S(n-2)$.
2. Determine constants a and b such that $T(n) = an + b$ solves the inhomogeneous recurrence T.
3. Find a solution for $T(n)$ which respects the given base cases.

Solution

1. The characteristic equation is $r^2 - 3r - 10 = 0$ which factors as $(r-5)(r+2) = 0$ so that the general solution to S is $S(n) = \alpha \cdot 5^n + \beta \cdot (-2)^n$.
2. We must have

$$\begin{aligned} an + b &= 3\left(a(n-1) + b\right) + 10\left(a(n-2) + b\right) + n - 2 \\ &= 3an - 3a + 3b + 10an - 20a + 10b + n - 2 \\ &= (13a + 1)\, n - 23a + 13b - 2 \end{aligned}$$

   so that

$$\begin{aligned} a &= 13a + 1 \\ b &= -23a + 13b - 2 \end{aligned}$$

   and therefore

$$\begin{aligned} a &= -\frac{1}{12} \\ b &= \frac{1}{144}. \end{aligned}$$

3. We know that $T(n) = \alpha \cdot 5^n + \beta \cdot (-2)^n - \frac{1}{12}n + \frac{1}{144}$ is the general solution to $T(n)$. With $n = 0$, we have the condition that $0 = \alpha + \beta + \frac{1}{144}$, so that $\alpha + \beta = -\frac{1}{144}$. With $n = 1$, we have the condition that $0 = 5\alpha - 2\beta - \frac{1}{12} + \frac{1}{144}$, so that $5\alpha - 2\beta = \frac{11}{144}$. Solving this system, we see that $\alpha = \frac{9}{1008}$ and $\beta = -\frac{1}{63}$. Our solution to T which respects the base cases is therefore
$$T(n) = \frac{9}{1008} \cdot 5^n - \frac{1}{63} \cdot (-2)^n - \frac{1}{12}n + \frac{1}{144}.$$

5.2 Asymptotic Solutions

For each recurrence below, find a simple function $f(n)$ so that $T(n) = \Theta(f(n))$.

1. $T(n) = T(n-1) + \frac{1}{n}$
2. $T(n) = T(n/2) + n$
3. $T(n) = 2T(n/2) + \sqrt{n}$
4. $T(n) = T(n/2) + T(n/3) + T(n/6) + n$

Solutions

1. This is the harmonic series; $T(n) = H_n = \Theta(\log n)$.
2. $T(n) = n + \frac{n}{2} + \frac{n}{4} + \cdots = n\left(1 + \frac{1}{2} + \frac{1}{4} + \cdots\right) = \Theta(n)$.
3. Using the recursion tree method, the kth level has 2^k nodes, each of which contributes $\sqrt{n/2^k}$ of work. Because there

are $d = \lg n$ levels in the recursion tree, we have

$$\begin{aligned}
T(n) &= \sum_{k=0}^{d} 2^k \sqrt{\frac{n}{2^k}} \\
&= \sqrt{n} \cdot \sum_{k=0}^{d} \sqrt{2}^k \\
&= \sqrt{n} \cdot \frac{\sqrt{2}^{d+1} - 1}{\sqrt{2} - 1} \\
&= \sqrt{n} \cdot \frac{\sqrt{2} \cdot 2^{\frac{1}{2}d} - 1}{\sqrt{2} - 1} \\
&= \sqrt{n} \cdot \frac{\sqrt{2} \cdot 2^{\frac{1}{2}\lg n} - 1}{\sqrt{2} - 1} \\
&= \sqrt{n} \cdot \frac{\sqrt{2} \cdot 2^{\lg \sqrt{n}} - 1}{\sqrt{2} - 1} \\
&= \sqrt{n} \cdot \frac{\sqrt{2} \cdot \sqrt{n} - 1}{\sqrt{2} - 1} \\
&= \frac{\sqrt{2}n - \sqrt{n}}{\sqrt{2} - 1} \\
&= \Theta(n).
\end{aligned}$$

4. Again, using the recursion tree method, we observe that each level of the recursion tree contributes n to $T(n)$. (This can be proved by induction and happens because $n/2 + n/3 + n/6 = n$.) Because the recursion tree has $d = \lg n$ levels, we have that $T(n) = \Theta(n \log n)$.

6 Probability

Suppose you roll a fair, six-sided die n times.

1. What is the probability that the sum of the numbers rolled is divisible by 3?

2. What is the probability that you see the same number twice in a row?

Solution

1. If $n = 0$, the probability is 1. If $n \geq 1$, then regardless of the first $n-1$ rolls, on the final roll exactly 2 out of the 6 numbers $\{1, 2, 3, 4, 5, 6\}$ will result in a sum which is divisible by 3. Therefore if $n \geq 1$, the probability that the sum is divisible by 3 is $\frac{1}{3}$.

2. If $n = 0$, the probability is 1. We assume that $n \geq 1$. Let A be the event that we see the same number rolled twice in a row. In this case, it is easier to consider the complentary event $B = \overline{A}$ that we do not see the same number rolled twice in a row. Because each outcome is equally likely, $\Pr(B) = \frac{|B|}{|\Omega|} = \frac{|B|}{6^n}$. To count all sequences of die rolls in B, observe that there are 6 possibilities for the first roll, and 5 possibilities for each subsequent roll: anything will work except the number just rolled. Therefore $|B| = 6 \cdot 5^{n-1}$. It follows that $\Pr(B) = \frac{|B|}{6^n} = \frac{6 \cdot 5^{n-1}}{6^n} = \left(\frac{5}{6}\right)^{n-1}$ and therefore $\Pr(A) = 1 - \Pr(B) = 1 - \left(\frac{5}{6}\right)^{n-1}$.