CSTBC Exam 2
Due: July 30, 2007, 11:59pm

July 21, 2007

This exam is open notes/open lecture and covers material from lectures 7-17. You are welcome to use any of the course material linked from the CSTBC website. You should not use other reference materials. Please send me your solutions to the exam via email. If you have any questions, please ask me.

1 Pirate's Dinner

After distributing their treasure, our n pirates from Exam 1 have worked up an appetite. The pirate ship's cafeteria offers three different, non-overlapping dinner times. As the strongest pirate, you are in charge of assigning each pirate to one of the three dinner slots. Unfortunately, not all pirates get along with each other. You have a list of k pairs of pirates that have fought each other in the past. Prove that you can assign pirates to dinners so that at most $k/3$ of the troublesome pairs eat dinner at the same time.

2 Hamiltonian Paths in Tournaments

In Lecture 12, we introduce the concept of a tournament. Prove that if T is a tournament on n vertices, then T contains a Hamiltonian path.

3 A Puzzle

Let $k \geq 0$ be an integer and let $n = 2^k$. Suppose you are given a square which is divided into n^2 smaller squares. The small squares are all colored blue, except that one special square is colored red. You are also given access to green game pieces which can be placed on the board. Each game piece comes in an 'L' shape and covers three of the smaller squares. Show that you can place the game pieces on the board in such a way that the pieces do not overlap and cover all the blue squares, leaving the red square uncovered. An example with $n = 8$ appears below.
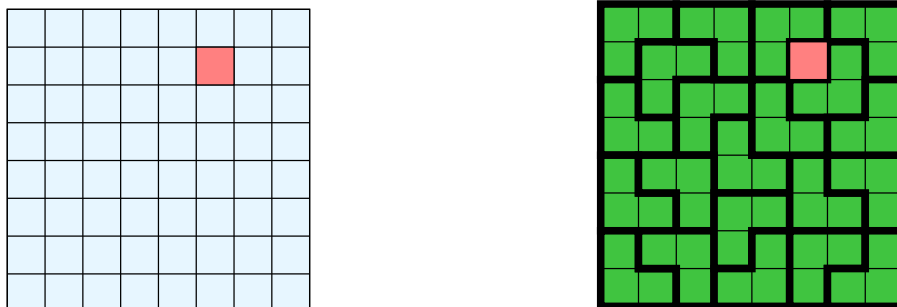

Figure 1: A puzzle and solution with $n = 8$

4 Graphs with (almost) unique degrees

In Lecture 3, we saw that a graph G with at least two vertices must contain two vertices of the same degree. Let G be a graph that contains a vertex u such that no two vertices in $V(G) - \{u\}$ have the same degree. What can you say about the degree of u?

5 Recurrences

5.1 An Exact Solution

In this problem, we will solve a linear homogeneous recurrence (see Lecture 16). Let

$$T(n) = \begin{cases} 0 & n \in \{0, 1\} \\ 3T(n-1) + 10T(n-2) + n - 2 & n \geq 2 \end{cases}.$$

1. Find the general solution to the homogeneous recurrence $S(n) = 3S(n-1) + 10S(n-2)$.

2. Determine constants a and b such that $T(n) = an + b$ solves the inhomogeneous recurrence T.

3. Find a solution for $T(n)$ which respects the given base cases.

5.2 Asymptotic Solutions

For each recurrence below, find a simple function $f(n)$ so that $T(n) = \Theta(f(n))$.

1. $T(n) = T(n-1) + \frac{1}{n}$

2. $T(n) = T(n/2) + n$

3. $T(n) = 2T(n/2) + \sqrt{n}$

4. $T(n) = T(n/2) + T(n/3) + T(n/6) + n$

6 Probability

Suppose you roll a fair, six-sided die n times.

1. What is the probability that the sum of the numbers rolled is divisible by 3?

2. What is the probability that you see the same number twice in a row?