

Turán Numbers of Ordered Tight Hyperpaths

John P. Bright, Kevin G. Milans, Jackson Porter

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Abstract

An *ordered hypergraph* is a hypergraph G whose vertex set $V(G)$ is linearly ordered. We find the Turán numbers for the r -uniform s -vertex tight path $\vec{P}_s^{(r)}$ (with vertices in the natural order) exactly when $r \leq s < 2r$ and n is even; our results imply $\vec{\text{ex}}(n, \vec{P}_s^{(r)}) = (1 - \frac{1}{2s-r} + o(1))\binom{n}{r}$ when $r \leq s < 2r$. When $r \geq 2s$, the asymptotics of $\vec{\text{ex}}(n, \vec{P}_s^{(r)})$ remain open. For $r = 3$, we give a construction of an r -uniform n -vertex hypergraph not containing $\vec{P}_s^{(r)}$ which we conjecture to be asymptotically extremal.

1 Introduction

The *Turán number* of an r -uniform hypergraph H , denoted $\text{ex}(n, H)$, is the maximum number of edges in an r -uniform n -vertex graph G that does not contain H as a subgraph. Bounding Turán numbers is a classical problem in extremal graph theory. The best known general bounds on the Turán numbers of the r -uniform s -vertex complete hypergraph $K_s^{(r)}$ are $(1 - (\frac{r-1}{s-1})^{r-1} - o(1))\binom{n}{r} \leq \text{ex}(n, K_s^{(r)}) \leq (1 - (\frac{s-1}{r-1})^{-1} + o(1))\binom{n}{r}$, with lower bound due to Sidorenko [9] and upper bound due to de Caen [1].

An *ordered hypergraph* is a hypergraph G whose vertices are linearly ordered. For an ordered hypergraph G , the *underlying hypergraph* is the ordinary hypergraph obtained from G by discarding the order on $V(G)$. For vertices u and v in an ordered hypergraph G , we write $u <_G v$, or $u < v$ when G is clear from context, if u appears before v in the ordering of $V(G)$. If G and H are ordered hypergraphs, then G is a *subgraph* of H , denoted $G \subseteq H$, if there is an injection $f: V(G) \rightarrow V(H)$ such that $u <_G v$ if and only if $f(u) <_H f(v)$ and $e \in E(G)$ implies $f(e) \in E(H)$, where $f(e) = \{f(v) : v \in e\}$. When H is an r -uniform ordered hypergraph, we use $\vec{\text{ex}}(n, H)$ to denote the analogous *ordered Turán number*, so that $\vec{\text{ex}}(n, H)$ is the maximum number of edges in an r -uniform n -vertex ordered hypergraph not containing H as a subgraph.

For graphs, ordered Turán numbers behave somewhat analogously to ordinary Turán numbers. The *interval chromatic number* of an ordered graph G , denoted $\chi_i(G)$, is the minimum k such that $V(G)$ can be partitioned into k intervals, each of which is an independent set. Although computing the chromatic number of an ordinary graph is NP-hard, an easy greedy algorithm computes $\chi_i(G)$ for an ordered graph G . Pach and Tardos [8] obtained an ordered analogue of the Erdős–Stone Theorem, showing that for each ordered graph H , we have $\vec{\text{ex}}(n, H) = (1 - \frac{1}{\chi_i(H)-1} + o(1))\binom{n}{2}$. Like the Erdős–Stone Theorem, this establishes the Turán numbers asymptotically for each ordered graph G with $\chi_i(G) > 2$. It is therefore natural to focus on ordered graphs G with $\chi_i(G) = 2$ and ordered hypergraphs.

A graph G is a *forest* if G has no cycles. Using classical Turán Theory, it is straightforward to show that $\vec{\text{ex}}(n, G) \geq \Omega(n^{1+\varepsilon})$ for some positive ε unless G is an ordered forest with $\chi_i(G) = 2$. Pach and Tardos [8] conjectured that if G is an ordered forest with $\chi_i(G) = 2$, then $\vec{\text{ex}}(n, G) \leq n(\log n)^{O(1)}$. Korándi, Tardos, Tomon, and Weidert [7] made progress on the conjecture by proving that $\vec{\text{ex}}(n, G) \leq n^{1+o(1)}$ when G is an ordered forest with $\chi_i(G) = 2$. For a family of ordered graphs \mathcal{G} , we define $\vec{\text{ex}}(n, \mathcal{G})$ to be the maximum number of edges in an n -vertex ordered graph that contains no member of \mathcal{G} as a subgraph. A *bordered cycle* is an ordered graph G whose underlying graph is a cycle, whose ordering has intervals X and Y with $X < Y$ such that each edge in G has an endpoint in X and an endpoint in Y (implying $\chi_i(G) \leq 2$), and contains the edge joining $\min X$ and $\max Y$ and the edge joining $\max X$ and $\min Y$. Győri, Korándi, Methuku, Tomon,

Tompkins, and Vizer [6] proved that $\vec{\text{ex}}(n, \mathcal{G}_k) = \Theta(n^{1+1/k})$, where \mathcal{G}_k is the family of bordered cycles on at most $2k$ vertices.

The r -uniform s -vertex *natural path*, denoted $\vec{P}_s^{(r)}$, has vertex set $\{v_1, \dots, v_s\}$ in the natural order $v_1 < \dots < v_s$ with $E(\vec{P}_s^{(r)})$ consisting of all intervals of size r . The underlying hypergraph of $\vec{P}_s^{(r)}$ is the well-known *tight path* $P_s^{(r)}$. The *length* of a path is the number of edges in the path, and so both $\vec{P}_s^{(r)}$ and $P_s^{(r)}$ have length $s - r + 1$. A special case of a conjecture by Kalai [2] states that for $n \geq r \geq 2$ and $s \geq r$, we have $\text{ex}(n, P_s^{(r)}) \leq \frac{s-r}{r} \binom{n}{r-1}$, which remains open. Füredi, Jiang, Kostochka, Mubayi, and Verstraëte [3] proved that $\text{ex}(n, P_6^{(3)}) = \binom{n}{2}$ for $n \geq 5$. In a later paper [4], the same authors proved that if $s \geq r$, then $\text{ex}(n, P_s^{(r)}) \leq \frac{s-r}{2} \binom{n}{r-1}$ when r is even and $\text{ex}(n, P_s^{(r)}) \leq \frac{1}{2}(s - r + 1 + \lfloor \frac{s-r}{r} \rfloor) \binom{n}{r-1}$ when r is odd.

Few results on Turán numbers of ordered hypergraphs are known. In classical Turán theory, an r -uniform hypergraph G satisfies $\text{ex}(n, G) = o(n^r)$ if and only if G is r -partite, meaning that there is a partition of $V(G)$ into r parts such that each edge in G has one vertex in each part. The analogous statement holds for ordered hypergraphs: an r -uniform ordered hypergraph G satisfies $\vec{\text{ex}}(n, G) = o(n^r)$ if and only if G is r -interval-partite, meaning that $V(G)$ can be partitioned into r intervals such that each edge in G has one vertex in each interval.

For $s > r$, the natural paths $\vec{P}_s^{(r)}$ are not r -interval-partite, and so $\vec{\text{ex}}(n, \vec{P}_s^{(r)}) \geq \Omega(n^r)$. The vertices of a tight path can be arranged in a different order to give an ordered r -interval-partite hypergraph. The r -uniform s -vertex *crossing path*, denoted $\vec{Q}_s^{(r)}$, is a tight path whose vertices are ordered as follows. Arrange the s vertices in an a grid with r rows R_1, \dots, R_r and $\lceil s/r \rceil$ columns such that any empty cells form a suffix of the last column. Let $t = \lceil s/r \rceil$, and let C_1, \dots, C_t be the columns of the grid. The ordering on the vertices of $\vec{Q}_s^{(r)}$ satisfies $R_1 < \dots < R_r$, where the vertices in each R_i are ordered from C_1 to C_t (or C_{t-1} if R_i has no vertex in row C_t). The edges of $\vec{Q}_s^{(r)}$ are the intervals of size r in the alternative vertex ordering such that $C_1 < \dots < C_t$, where the vertices in each C_j are ordered from R_1 to R_r (or, in the case of C_t , from R_1 to the last occupied row). Since each edge in $\vec{Q}_s^{(r)}$ has one vertex in each row and $R_1 < \dots < R_r$, it follows that $\vec{Q}_s^{(r)}$ is r -interval-partite. Füredi, Jiang, Kostochka, Mubayi, and Verstraëte [5] proved that $\vec{\text{ex}}(n, \vec{Q}_s^{(r)})$ equals $\binom{n}{r} - \binom{n-(s-r)}{r}$ when $r \leq s \leq 2r$ and is $\Theta(n^{r-1} \log n)$ when $s > 2r$. Since $\binom{n}{r} - \binom{n-(s-r)}{r} = (1 + o(1))r(s-r)n^{r-1}$, it follows that always $\vec{\text{ex}}(n, \vec{Q}_s^{(r)}) = O(n^{r-1} \log(n))$. A hypergraph F is a *forest* if the edges of F can be ordered as e_1, \dots, e_m such that for each i , the edge e_i is the union of a subset of an earlier edge e_j with $j < i$ and vertices that are not contained in any edge in $\{e_1, \dots, e_{i-1}\}$. In classical Turán theory, we have $\text{ex}(n, F) \leq O(n^{r-1})$ for each r -uniform forest F . Generalizing the Pach-Tardos conjecture, Füredi et al [5] conjectured that $\vec{\text{ex}}(n, F) = O(n^{r-1} \cdot \text{polylog}(n))$ when F is an r -uniform r -interval-partite forest.

We are interested in $\vec{\text{ex}}(n, \vec{P}_s^{(r)})$. In Section 2, we obtain $\vec{\text{ex}}(n, \vec{P}_s^{(r)})$ exactly when $s \leq 2r - 1$ and n is even, implying that $\vec{\text{ex}}(n, \vec{P}_s^{(r)}) = (1 - \frac{1}{2^{s-r}} + o(1)) \binom{n}{r}$ when $r \leq s \leq 2r - 1$. When $s \geq 2r$, determining the asymptotics of $\vec{\text{ex}}(n, \vec{P}_s^{(r)})$ remains open. When r divides s , our fractional results in Section 3 imply $\vec{\text{ex}}(n, \vec{P}_s^{(r)}) \leq (1 - (\frac{r}{s})^r + o(1)) \binom{n}{r}$. When $r - 1$ divides $s - 1$, partitioning an interval of n vertices into $(s-1)/(r-1)$ parts of equal size and removing edges with all vertices in a single part shows that $\vec{\text{ex}}(n, \vec{P}_s^{(r)}) \geq (1 - (\frac{r-1}{s-1})^{r-1} - o(1)) \binom{n}{r}$, matching the Sidorenko lower bound on $\text{ex}(n, \vec{K}_s^{(r)})$ even though $\vec{\text{ex}}(n, \vec{P}_s^{(r)}) \leq \vec{\text{ex}}(n, \vec{K}_s^{(r)}) = \text{ex}(n, K_s^{(r)})$. When r and s do not have convenient divisibility relationships, obtaining bounds on $\vec{\text{ex}}(n, \vec{P}_s^{(r)})$ may involve additional subtleties. Sidorenko's lower bound on $\text{ex}(n, \vec{K}_s^{(r)})$ holds for general r and s ; in fact, the argument shows that $\vec{\text{ex}}(n, \vec{C}_s^{(r)}) \geq (1 - (\frac{r-1}{s-1})^{r-1} - o(1)) \binom{n}{r}$, where $\vec{C}_s^{(r)}$ is the r -uniform s -vertex ordered tight cycle, with vertex set $\{v_0, \dots, v_{s-1}\}$ in the natural order and edge set $\{e_0, \dots, e_{s-1}\}$, where $e_j = \{v_j, \dots, v_{j+r-1}\}$ (subscript arithmetic modulo s).

We study the case $r = 3$ in Section 4. When s is odd, we have that $r - 1$ divides $s - 1$ and the same construction as above gives $\vec{\text{ex}}(n, \vec{P}_s^{(3)}) \geq (1 - \frac{4}{(s-1)^2} - o(1)) \binom{n}{3}$. When s is even, we give a construction that improves the lower bound to $\vec{\text{ex}}(n, \vec{P}_s^{(3)}) \geq (1 - \frac{4}{s(s-2)} - o(1)) \binom{n}{3}$. We conjecture that these bounds are asymptotically sharp. An ordered hypergraph G is *monotone* if, for each edge $\{u, v, w\} \in E(G)$ with $u < v < w$, we have $\ell(uv) \leq \ell(vw)$, where $\ell(xy)$ is the length of a longest ordered tight path whose last two

vertices are x followed by y (see Section 4 for an equivalent, but perhaps more natural, formulation). As some partial evidence for this conjecture, we show that if s is odd and G is a monotone n -vertex ordered hypergraph not containing $\vec{P}_s^{(r)}$, then $|E(G)| \leq (1 - \frac{4}{(s-1)^2} + o(1))\binom{n}{3}$. The first unresolved case is that of $P_6^{(3)}$, with best known bounds $(\frac{5}{6} - o(1))\binom{n}{3} \leq \vec{\text{ex}}(n, P_6^{(3)}) \leq (\frac{7}{8} + o(1))\binom{n}{3}$.

2 Exact Results for Short Paths

In this section, our aim is to establish $\vec{\text{ex}}(n, \vec{P}_s^{(r)})$ exactly when $r \leq s \leq 2r - 1$ and n is even. If $G \subseteq \vec{K}_n^{(r)}$ and G does not contain H , then each copy of H in $\vec{K}_n^{(r)}$ has some edge in \overline{G} . An H -transversal in $\vec{K}_n^{(r)}$ is a graph $G' \subseteq \vec{K}_n^{(r)}$ such that every copy of H in $\vec{K}_n^{(r)}$ has at least one edge in G' . The transversal number of H , denoted $\vec{\tau}(n, H)$, is the minimum number of edges in an H -transversal. Note that $\vec{\text{ex}}(n, H) + \vec{\tau}(n, H) = |E(\vec{K}_n^{(r)})| = \binom{n}{r}$.

We use $[n]$ for the vertex set of $\vec{K}_n^{(r)}$. For vertex sets A and B in an ordered graph G , we write $A < B$ if $a < b$ for all $a \in A$ and $b \in B$. The reflection of a vertex u in $\vec{K}_n^{(r)}$ is the vertex $n + 1 - u$. An interval partition of $\vec{K}_n^{(r)}$ is a list of disjoint vertex subsets (X_1, \dots, X_k) whose union is $[n]$ such that each X_i is an interval in $[n]$ and $X_i < X_j$ when $i < j$. A set of vertices S is m -left-biased if $\vec{K}_n^{(r)}$ has an interval partition (X, Y, Z) such that $|X| = |Z|$, $|X \cap S| = m$, and $|Z \cap S| = 0$. Similarly, S is m -right-biased if $\vec{K}_n^{(r)}$ has an interval partition (X, Y, Z) such that $|X| = |Z|$, $|X \cap S| = 0$, and $|Z \cap S| = m$. We say that S is m -biased if S is m -left-biased or m -right-biased. Let $h(n, t, m)$ be the number of t -sets that are m -left-biased in $\vec{K}_n^{(r)}$. Note that for even n , summing the m -left-biased t -sets whose m th vertex is at index k shows that $h(n, t, m) = \sum_{k=m}^{n/2} \binom{k-1}{m-1} \binom{n-2k}{t-m}$ when $1 \leq m \leq t$.

Our next theorem gives a lower bound on $\vec{\text{ex}}(n, \vec{P}_s^{(r)})$ by constructing an $\vec{P}_s^{(r)}$ -transversal when $r \leq s \leq 2r - 1$. In fact, we construct a $\vec{L}P_s^{(r)}$ -transversal, where $\vec{L}P_s^{(r)}$ is the loose path obtained from $\vec{P}_s^{(r)}$ by removing all but the first and last edges. Note that the two edges in $\vec{L}P_s^{(r)}$ intersect when $r < s \leq 2r - 1$.

Theorem 1. *Let n be even, let $r \leq s \leq 2r - 1$, and let $m = |E(\vec{P}_s^{(r)})| = s - r + 1$. We have $\vec{\tau}(n, \vec{P}_s^{(r)}) \leq \vec{\tau}(n, \vec{L}P_s^{(r)}) \leq 2h(n, r, m) + h(n, r - 1, m)$.*

Proof. The condition $r \leq s \leq 2r - 1$ translates to $1 \leq m \leq r$. Let G be the subgraph of $\vec{K}_n^{(r)}$ such that $E(G) = E_1 \cup E_2$, where E_1 is the family of r -sets that are m -biased and E_2 is the family of r -sets whose m th and last vertices are reflections of one another. Since $m > 0$, the m -biased r -sets are the disjoint union of the m -left-biased r -sets and the m -right-biased r -sets, both of which have size $h(n, r, m)$, and so $|E_1| = 2h(n, r, m)$. Also, when $m < r$, removing the last vertex from an r -set whose m th and last points are reflections of one another gives an $(r - 1)$ -set that is m -left-biased and conversely, and so $|E_2| = h(n, r - 1, m)$. (Note that when $r = m$, we have $h(n, r - 1, m) = 0$.)

It remains to show that every copy of $\vec{L}P_s^{(r)}$ in $\vec{K}_n^{(r)}$ has an edge in G . Let Q be such a copy, and let (X, Y, Z) be the interval partition of $\vec{K}_n^{(r)}$ that minimizes $|X|$ subject to $|X| = |Z|$ and $\max\{|X \cap V(Q)|, |Z \cap V(Q)|\} \geq m$. Such a partition exists, or else Q has at most $m - 1$ vertices in both the left and right halves of $\vec{K}_n^{(r)}$, which would imply $s = |V(Q)| \leq 2(m - 1) = 2(s - r)$, contradicting $s \leq 2r - 1$. Let $u = |X|$.

Suppose first that $|X \cap V(Q)| = |Z \cap V(Q)| = m$. In this case, it must be that both u and its reflection $n + 1 - u$ are vertices in Q . Note that $s = r + (m - 1)$. Deleting the last $m - 1$ vertices in Q gives the first edge $e \in E(Q)$ whose m th vertex is u and whose last vertex is $n + 1 - u$, implying that $e \in E_2 \subseteq E(G)$.

Otherwise, one of $\{|X \cap V(Q)|, |Z \cap V(Q)|\}$ equals m and the other is at most $m - 1$. We show that Q has an edge in E_1 . Suppose that $|X \cap V(Q)| = m$ and $|Z \cap V(Q)| \leq m - 1$. Let e be the first edge in Q (which is obtained by deleting the last $m - 1$ vertices of Q). Since $s \geq m + (m - 1)$, none of the deleted vertices are in X . It follows that $|X \cap e| = m$ and $|Z \cap e| = 0$. So e is m -left-biased, and therefore $e \in E_1 \subseteq E(G)$. If instead $|X \cap V(Q)| \leq m - 1$ and $|Z \cap V(Q)| = m$, then a similar argument shows that the last edge e in Q is m -right-biased, also implying $e \in E_1 \subseteq E(G)$. \square

Our next theorem obtains a large family of edge-disjoint copies of $\vec{P}_s^{(r)}$. For an r -uniform ordered hypergraph H , the H -packing number, denoted $\vec{\nu}(n, H)$ is the maximum size of an edge-disjoint family of copies of H in $\vec{K}_n^{(r)}$. Clearly, $\vec{\nu}(n, H) \leq \vec{\tau}(n, H)$.

Theorem 2. *Let n be even, let $r \leq s \leq 2r - 1$, and let $m = |E(\vec{P}_s^{(r)})| = s - r + 1$. We have $\vec{\nu}(n, \vec{P}_s^{(r)}) \geq 2h(n, r, m) + h(n, r - 1, m)$.*

Proof. As in Theorem 1, let G be the subgraph of $\vec{K}_n^{(r)}$ with $E(G) = E_1 \cup E_2$, where E_1 is the family of r -sets that are m -biased and E_2 is the family of r -sets whose m th vertex and last vertex are reflections of one another. For each $e \in E(G)$, we construct a copy Q_e of $\vec{P}_s^{(r)}$ such that the family of paths $\{Q_e : e \in E(G)\}$ is edge-disjoint.

Let $e \in E(G)$. We construct Q_e as follows. Note that every edge in G is $(m - 1)$ -biased. The *canonical interval partition* of e is the interval partition (X, Y, Z) of $\vec{K}_n^{(r)}$ that maximizes $|X|$ subject to $|X| = |Z|$, $\max\{|X \cap e|, |Z \cap e|\} = m - 1$, and $\min\{|X \cap e|, |Z \cap e|\} = 0$. Note that by maximality of $|X|$, it follows that either $\min Y$ or $\max Y$ is a vertex in e , and so the canonical interval partition of e is also the interval partition (X, Y, Z) of $\vec{K}_n^{(r)}$ that minimizes $|Y|$ subject to $|X| = |Z|$ and $|Y \cap e| = r - (m - 1) = 2r - s$. The *translation* of a set of vertices $S \subseteq V(\vec{K}_n^{(r)})$ by a constant c , denoted $S + c$, is the set $\{u + c : u \in S\}$. If $|X \cap e| = m - 1$, then we take $V(Q_e)$ to be the union of e and the translation $(X \cap e) + (n - |Z|)$. Otherwise if $|Z \cap e| = m - 1$, then we take $V(Q_e)$ to be the union of e and the translation $(Z \cap e) - (n - |X|)$. In both cases, $|V(Q_e)| = r + m - 1 = s$, and so Q_e is a copy of $\vec{P}_s^{(r)}$. The *core* of Q_e is the set of $2r - s$ vertices in $V(Q_e)$ that belong to every edge in Q_e . Since $2r - s \geq 1$, the core of Q_e is non-empty. Also, since $2r - s = s - 2(m - 1)$ and $|X \cap V(Q_e)| = |Z \cap V(Q_e)| = m - 1$, it follows that the core of Q_e equals $Y \cap V(Q_e)$. Because the core of Q_e consists of the $2r - s$ vertices in Q_e that are closest to the center of $V(\vec{K}_n^{(r)})$, given any edge in Q_e we can identify the core of Q_e . Moreover, since the canonical interval partition of e is the partition (X, Y, Z) given by minimizing $|Y|$ subject to $|X| = |Z|$ and $|Y \cap e| = 2r - s$, each edge in Q_e also determines the canonical interval partition of e .

We show that given an edge f in some path Q_e in the family $\{Q_e : e \in E(G)\}$, we can determine the edge $e \in E(G)$ that generates Q_e . It follows that the family is edge-disjoint. Let f be an edge in one of the paths in our collection, and recall that f determines the canonical interval partition (X, Y, Z) of the generating edge e . It follows that $V(Q_e)$ is the union of f and the translations $(X \cap f) + (n - |Z|)$ and $(Z \cap f) - (n - |X|)$. Note that the edge $e \in E(G)$ that generates Q_e must be the first or last edge in Q_e . We identify e as follows. If $\min Y$ and $\max Y$ are in $V(Q_e)$, then $e \in E_2$ and e is the first edge in Q_e . If $\min Y$ is in $V(Q_e)$ but $\max Y$ is not, then $e \in E_1$ and e is also the first edge in Q_e . Otherwise, $\max Y$ is in $V(Q_e)$ and $\min Y$ is not, in which case $e \in E_1$ and e is the last edge in Q_e . \square

The theorems give exact results on $\vec{\nu}(n, \vec{P}_s^{(r)})$, $\vec{\tau}(n, \vec{P}_s^{(r)})$, and $\text{ex}(n, \vec{P}_s^{(r)})$ when $r \leq s \leq 2r - 1$.

Corollary 3. *Let n be even, let $r \leq s \leq 2r - 1$, and let $m = |E(\vec{P}_s^{(r)})| = s - r + 1$. We have that each parameter in $\{\vec{\nu}(n, \vec{P}_s^{(r)}), \vec{\nu}(n, \vec{L}\vec{P}_s^{(r)}), \vec{\tau}(n, \vec{P}_s^{(r)}), \vec{\tau}(n, \vec{L}\vec{P}_s^{(r)})\}$ equals $2h(n, r, m) + h(n, r - 1, m)$, and $2h(n, r, m) + h(n, r - 1, m) = \frac{1}{2^{m-1}} \binom{n}{r} + O(n^{r-1})$. Therefore $\vec{\text{ex}}(n, \vec{L}\vec{P}_s^{(r)}) = \vec{\text{ex}}(n, \vec{P}_s^{(r)}) = \binom{n}{r} - 2h(n, r, m) - h(n, r - 1, m) = (1 - \frac{1}{2^{m-1}}) \binom{n}{r} + O(n^{r-1})$.*

Proof. Clearly $\vec{\nu}(n, \vec{P}_s^{(r)}) \leq \vec{\nu}(n, \vec{L}\vec{P}_s^{(r)})$, $\vec{\tau}(n, \vec{P}_s^{(r)}) \leq \vec{\tau}(n, \vec{L}\vec{P}_s^{(r)})$. By Theorem 2 and Theorem 1 respectively, we have $\vec{\nu}(n, \vec{P}_s^{(r)}) \geq 2h(n, r, m) + h(n, r - 1, m)$ and $\vec{\tau}(n, \vec{L}\vec{P}_s^{(r)}) \leq 2h(n, r, m) + h(n, r - 1, m)$ and the exact results on $\vec{\nu}$, $\vec{\tau}$, and $\vec{\text{ex}}$ follow.

For the asymptotic results, it suffices to show that $h(n, r, m) = (1/2^m) \binom{n}{r} + O(n^{r-1})$. Recall that $h(n, r, m)$ is the number of m -left-biased r -sets. An r -set R is *degenerate* if R contains a vertex u and its reflection $n + 1 - u$, and R is *typical* if it is not degenerate. Let A be the family of degenerate r -sets, and let B be the family of typical r -sets. Note that $|A| \leq (r - 1) \binom{n}{r-1} = O(n^{r-1})$, since choosing $r - 1$ vertices and a vertex to reflect determines a degenerate r -set. Note that $|B| = 2^r \binom{n/2}{r}$, since each r -set in B is generated by choosing r of the reflection pairs, and then selecting a vertex from each chosen pair. The m -left-biased sets in B are obtained by choosing the left vertex from the m outermost reflection pairs,

and so $2^{r-m} \binom{n/2}{r}$ sets in B are m -left-biased. Let C be the family of m -left-biased r -sets. We compute $h(n, r, m) = |C| = |C \cap A| + |C \cap B| = O(n^{r-1}) + 2^{r-m} \binom{n/2}{r} = O(n^{r-1}) + \frac{1}{2^m} \binom{n}{r}$. \square

3 Fractional Variants

The transversal and packing numbers from Section 2 have fractional variants. For an ordered hypergraph H , a *fractional transversal* of the copies of H in $\vec{K}_n^{(r)}$ is a function w that assigns non-negative weights to each edge in $\vec{K}_n^{(r)}$ such that $\sum_{e \in E(H')} w(e) \geq 1$ for each copy H' of H in $\vec{K}_n^{(r)}$. The *fractional transversal number* of H , denoted $\vec{\tau}^*(n, H)$, is the infimum, over all fractional transversals w , of the sum of $w(e)$ over all edges $e \in E(\vec{K}_n^{(r)})$. Standard compactness arguments show that the infimum is always achieved, and so we may replace infimum by minimum in the definition. Also, if $G' \subseteq \vec{K}_n^{(r)}$ and G' is an H -transversal, then the weight function w with $w(e) = 1$ for $e \in E(G')$ and $w(e) = 0$ for $e \notin E(G')$ is a fractional transversal, and therefore $\vec{\tau}^*(n, H) \leq \vec{\tau}(n, H)$.

Each fractional transversal of copies of H in $\vec{K}_n^{(r)}$ is a feasible solution to the linear program with variables $\{w(e) : e \in E(\vec{K}_n^{(r)})\}$ with objective to minimize $\sum_e w(e)$ subject to $\sum_{e \in E(H')} w(e) \geq 1$ for each copy H' of H in $\vec{K}_n^{(r)}$. The dual linear program has variables $\{w(H') : H' \text{ is a copy of } H \text{ in } \vec{K}_n^{(r)}\}$ with objective to maximize $\sum_{H'} w(H')$ subject to the constraints that, for each edge e in $\vec{K}_n^{(r)}$, the sum of $w(H')$ over all copies H' of H in $\vec{K}_n^{(r)}$ that contain e is at most 1. A feasible solution w to the dual linear program is called a *fractional H -packing*, and the *fractional H -packing number*, denoted $\vec{\nu}^*(n, H)$ is the value of this linear program. Since both the LP and its dual are clearly feasible, it follows from theory of linear programming that $\vec{\tau}^*(n, H) = \vec{\nu}^*(n, H)$. As before, standard compactness arguments show that a fractional H -packing with total weight $\vec{\nu}^*(n, H)$ exists, and it is clear that $\vec{\nu}(n, H) \leq \vec{\nu}^*(n, H)$. Hence

$$\vec{\nu}(n, H) \leq \vec{\nu}^*(n, H) = \vec{\tau}^*(n, H) \leq \vec{\tau}(n, H).$$

In this section, we show that $\vec{\nu}(n, \vec{P}_s^{(r)})$, $\vec{\nu}^*(n, \vec{P}_s^{(r)})$, and $\vec{\tau}^*(n, \vec{P}_s^{(r)})$ are all asymptotically $((\frac{r}{s})^r + o(1)) \binom{n}{r}$ when r divides s .

Proposition 4. *If r divides s , then $\vec{\nu}(n, \vec{P}_s^{(r)}) \geq ((\frac{r}{s})^r + o(1)) \binom{n}{r}$.*

Proof. We give a $\vec{P}_s^{(r)}$ -packing of the required size. Let $k = s/r$, and without loss of generality assume that k divides n . Let (X_1, \dots, X_k) be an interval partition of $\vec{K}_n^{(r)}$ into parts of equal size. For each edge $e \in E(\vec{K}_n^{(r)})$ with $e \subseteq X_1$, let P_e be the s -vertex path with vertex set $\bigcup_{j=0}^{k-1} (e + j|X_1|)$. Note that given any edge $e' \in E(P_e)$, it is easy to recover e , and it follows that $\{P_e : e \subseteq X_1\}$ is an edge-disjoint collection of copies of $\vec{P}_s^{(r)}$. Therefore $\vec{\nu}(n, \vec{P}_s^{(r)}) \geq \binom{|X_1|}{r} = \binom{n/k}{r} = (\frac{1}{k^r} + o(1)) \binom{n}{r}$. \square

Proposition 5. *If r divides s , then $\vec{\tau}^*(n, \vec{P}_s^{(r)}) \leq ((\frac{r}{s})^r + o(1)) \binom{n}{r}$.*

Proof. We give a fractional $\vec{P}_s^{(r)}$ -transversal. Let $k = s/r$. We may assume without loss of generality that k divides n . Let (X_1, \dots, X_k) be an interval partition of $\vec{K}_n^{(r)}$ into parts of equal size. Let w be the weight function with $w(e) = r/s$ if e is contained in a part in (X_1, \dots, X_k) and $w(e) = 0$ otherwise. Let P be a copy of $\vec{P}_s^{(r)}$ in $\vec{K}_n^{(r)}$. Note that at most $r - 1$ vertices in $V(P) \cap X_i$ begin an edge with weight zero, and P has at most $k(r - 1)$ such vertices. Therefore at least $s - k(r - 1)$ vertices in P begin an edge with positive weight. So the edges of P have total weight at least $(r/s)(s - k(r - 1))$, and this equals 1. It follows that w is a fractional $\vec{P}_s^{(r)}$ -transversal, and so $\vec{\tau}^*(n, \vec{P}_s^{(r)}) \leq \frac{r}{s} \cdot k \cdot \binom{|X_1|}{r} = \binom{n/k}{r} = (\frac{1}{k^r} + o(1)) \binom{n}{r}$. \square

If r divides s and the integer s/r also divides n , then Proposition 4 and Proposition 5 imply $\vec{\nu}(n, \vec{P}_s^{(r)}) = \vec{\tau}^*(n, \vec{P}_s^{(r)}) = \vec{\nu}^*(n, \vec{P}_s^{(r)}) = \binom{n/k}{r}$, where $k = s/r$.

Theorem 6. *If r divides s , then $\vec{\nu}(n, \vec{P}_s^{(r)})$, $\vec{\nu}^*(n, \vec{P}_s^{(r)})$, and $\vec{\tau}^*(n, \vec{P}_s^{(r)})$ are all asymptotically equal to $((\frac{r}{s})^r + o(1)) \binom{n}{r}$.*

The smallest path to which our argument in Section 2 does not apply is $P_{2r}^{(r)}$, when $s = 2r$. In this case, our fractional results imply $\vec{\tau}^*(n, P_{2r}^{(r)}) = (\frac{1}{2r} + o(1)) \binom{n}{r}$, but, at least in the case $r = 3$, we believe that $\vec{\tau}(n, P_{2r}^{(r)}) \gg \vec{\tau}^*(n, P_{2r}^{(r)})$. In particular, $\vec{\tau}^*(n, P_6^{(3)}) = (\frac{1}{8} + o(1)) \binom{n}{3}$ but we conjecture $\vec{\tau}(n, P_6^{(3)}) = (\frac{1}{6} + o(1)) \binom{n}{3}$.

4 The case $r = 3$

When $r = 3$, the ordered Turán numbers are equivalent to an edge-labeling problem on the ordered complete graph $\vec{K}_n^{(2)}$. A k -edge-labeling of $\vec{K}_n^{(2)}$ assigns to each pair uv a label in a linearly ordered set S with $|S| = k$. Let ϕ be a k -edge-labeling of $\vec{K}_n^{(2)}$. A triple of vertices $\{u, v, w\}$ with $u < v < w$ is *good* if $\phi(uv) < \phi(vw)$. For convenience, we write uvw for the triple $\{u, v, w\}$. A triple is *bad* if it is not good. Let $f(n, k)$ be the maximum, over all k -edge-labelings ϕ of $\vec{K}_n^{(2)}$, of the number of good triples. A k -edge-labeling ϕ of $\vec{K}_n^{(2)}$ is *optimal* if it has $f(n, k)$ good triples.

Proposition 7. *For $s \geq 3$, we have $\vec{\text{ex}}(n, P_s^{(3)}) = f(n, s - 2)$.*

Proof. First, we show $\vec{\text{ex}}(n, P_s^{(3)}) \geq f(n, s - 2)$. Let ϕ be an $(s - 2)$ -edge-labeling of $\vec{K}_n^{(2)}$ with $f(n, s - 2)$ good triples. Let G be the ordered 3-uniform hypergraph with vertex set $V(\vec{K}_n^{(2)})$ such that for $u < v < w$, we have $uvw \in E(G)$ if and only if $\phi(uv) < \phi(vw)$. We claim that G does not contain $P_s^{(3)}$. Indeed, if $v_1 \cdots v_s$ is a copy of $P_s^{(3)}$ in G , then $\phi(v_{i-1}v_i) < \phi(v_i v_{i+1})$ for $1 < i < s$ by definition of G . It follows that ϕ uses $s - 1$ distinct labels on the consecutive pairs of $v_1 \cdots v_s$, contradicting that ϕ is a $(s - 2)$ -edge-labeling. It follows that $\vec{\text{ex}}(n, P_s^{(3)}) \geq |E(G)| = f(n, s - 2)$.

Next, we show $\vec{\text{ex}}(n, P_s^{(3)}) \leq f(n, s - 2)$. Let G be a 3-uniform ordered hypergraph not containing $P_s^{(3)}$, and let ϕ be the edge-labeling on $V(G)$ by setting $\phi(uv)$, where $u < v$, equal to the length of a longest tight ordered path in G that ends in uv . Clearly, if $u < v < w$ and $uvw \in E(G)$, then we have $\phi(uv) < \phi(vw)$ since the edge uvw can be used to extend a longest ordered path ending in uv to a longest path of larger length ending in vw . Therefore ϕ has at least $|E(G)|$ good triples. Note that ϕ assigns each pair uv a value in the set $\{0, \dots, s - 3\}$, since every ordered tight path in G has at most $s - 1$ vertices and at most $s - 3$ edges. Since ϕ assigns each edge a value in $\{0, \dots, s - 3\}$, it follows that ϕ is an $(s - 2)$ -edge-labeling, and therefore $f(n, s - 2) \geq |E(G)| = \vec{\text{ex}}(n, P_s^{(3)})$. \square

Next, we give lower bound constructions for $f(n, k)$ which we conjecture to be asymptotically optimal. The construction is easiest to describe when k is odd. A labeling ϕ is *monotone* if, for all $u < v < w$, we have $\phi(uv) \leq \phi(vw)$.

Proposition 8. *If k is odd, then $f(n, k) \geq \left(1 - \frac{4}{(k+1)^2} + o(1)\right) \binom{n}{3}$.*

Proof. Let $t = (k + 1)/2$. Let (X_1, \dots, X_t) be an interval partition of $\vec{K}_n^{(2)}$ into t parts whose sizes differ by at most 1. For $u < v$ with $u \in X_i$ and $v \in X_j$, we set $\phi(uv) = i + j$. Clearly, the range of ϕ is contained in $\{2, \dots, 2t\}$, and so ϕ is a $(2t - 1)$ -edge-labeling. Note that $2t - 1 = k$.

Since ϕ is a monotone labeling, the only triples uvw with $u < v < w$ that are not good are those with $\phi(uv) = \phi(vw)$. Each such triple is contained in a part X_i for some i . It follows that the number of triples that are not good is asymptotically equal to $t \binom{n/t}{3}$, which is asymptotically equal to $\frac{1}{t^2} \binom{n}{3}$. The proposition follows. \square

Corollary 9. *If $s \geq 3$ and s is odd, then $\vec{\text{ex}}(n, P_s^{(3)}) \geq \left(1 - \frac{4}{(s-1)^2} + o(1)\right) \binom{n}{3}$.*

Applying Proposition 7 to the construction in Proposition 8 gives a graph G whose complement is the union of t complete graphs on t disjoint intervals of nearly equal size. The construction for even k is more subtle.

We first give our construction in terms of a general interval partition (X_1, \dots, X_k) of $[n]$ into k parts. Later, we specify the sizes of the parts. For a pair uv with $u < v$, $u \in X_i$, and $v \in X_j$, we define $\phi(uv)$ as follows. If $i = j$, then $\phi(uv) = i$. If $j - i \geq 2$, then we set $\phi(uv)$ so that $i < \phi(uv) < j$. Otherwise, $j = i + 1$. The *fractional index* of u in X_i is $(u + 1 - \min X_i)/|X_i|$. Note that the fractional index of u is a real number in $(0, 1]$. Let λ_u and λ_v be the fractional indices of u in X_i and v in X_{i+1} , respectively. We set $\phi(uv) = i$ if $\lambda_u + \lambda_v \leq 1$ and $\phi(uv) = i + 1$ if $\lambda_u + \lambda_v > 1$.

Lemma 10. *Let a, b, c be constants. If $|X_{i-1}| = (a + o(1))n$, $|X_i| = (b + o(1))n$, and $|X_{i+1}| = (c + o(1))n$, then the number of bad triples uvw with $v \in X_i$ is $[(a + b)b(b + c) + o(1)] \binom{n}{3}$.*

Proof. Let $v \in X_i$ and let λ_v be the fractional index of v in X_i . If uvw is a bad triple with $u < v < w$, then $u \in X_{i-1} \cup X_i$, $w \in X_i \cup X_{i+1}$, and $\phi(uv) = \phi(vw) = i$. For $u \in X_i$, we require only that u precede v , and there are $\lambda_v|X_i| - 1$ such vertices in X_i . For $u \in X_{i-1}$, the condition $\phi(uv) = i$ is equivalent to $\lambda_u + \lambda_v > 1$, where λ_u is the fractional index of $u \in X_{i-1}$. The number of $u \in X_{i-1}$ with $\lambda_u + \lambda_v > 1$ is $|X_{i-1}| - \lfloor (1 - \lambda_v)|X_{i-1}| \rfloor$ or $\lceil \lambda_v|X_{i-1}| \rceil$. It follows that the number of choices for u in a bad triple uvw with $u < v < w$ equals $\lambda_v(|X_{i-1}| + |X_i|) + O(1)$.

Similarly, the number of $w \in X_i$ that follow v is $(1 - \lambda_v)|X_i|$ and the number of $w \in X_{i+1}$ with $\lambda_v + \lambda_w \leq 1$ is $\lfloor (1 - \lambda_v)|X_{i+1}| \rfloor$. It follows that the number of choices for w in a bad triple uvw with $u < v < w$ equals $(1 - \lambda_v)(|X_i| + |X_{i+1}|) + O(1)$.

Multiplying the number of choices for u and the number of choices for w gives a total of $\lambda_v(1 - \lambda_v)(|X_{i-1}| + |X_i|)(|X_i| + |X_{i+1}|) + O(n)$ bad triples uvw with $u < v < w$. Suppose that $X_i = \{v_1, \dots, v_t\}$. Summing over all $v \in X_i$, the total number of bad triples is $O(n^2) + (|X_{i-1}| + |X_i|)(|X_i| + |X_{i+1}|) \frac{1}{t^2} \sum_{j=1}^t j(t - j)$, or $O(n^2) + (|X_{i-1}| + |X_i|)(|X_i| + |X_{i+1}|) \frac{1}{t^2} \binom{t+1}{3}$. Recalling that $t = |X_i| = (b + o(1))n$, the number of bad triples uvw with $u < v < w$ simplifies to $O(n^2) + (a + b)(b + c)b(\frac{1}{6} + o(1))n^3$ and the lemma follows. \square

Lemma 11. *If k is even, then $f(n, k) \geq \left(1 - \frac{4}{k(k+2)} - o(1)\right) \binom{n}{3}$.*

Proof. Suppose k is even, and let $t = k/2$. Let (Y_1, \dots, Y_t) and (Z_1, \dots, Z_{t+1}) be interval partitions of $V(\vec{K}_n^{(2)})$ into parts of nearly equal size, and let (X_1, \dots, X_k) be their common refinement. (Note that (Y_1, \dots, Y_t) is the partition used in our construction with $k - 1$ labels, and (Z_1, \dots, Z_{t+1}) is the partition used in our construction with $k + 1$ labels.) For $1 \leq j \leq k$, we set a_j equal to the limit of $|X_j|/n$ as $n \rightarrow \infty$. It is convenient to introduce $X_0 = X_{k+1} = \emptyset$ and $a_0 = a_{k+1} = 0$. When divided by n to normalize, the boundaries of (Y_1, \dots, Y_t) are $\frac{0}{t}, \frac{1}{t}, \dots, \frac{t}{t}$, and the boundaries of (Z_1, \dots, Z_{t+1}) are $\frac{0}{t+1}, \frac{1}{t+1}, \dots, \frac{t+1}{t+1}$. In the common refinement, these boundaries interleave and are thus $\frac{0}{t}, \frac{1}{t+1}, \frac{1}{t}, \frac{2}{t+1}, \frac{2}{t}, \dots, \frac{t}{t+1}, \frac{t}{t}$. It follows that for $0 \leq j \leq t$, we have $a_{2j} = \left(\frac{j}{t} - \frac{j}{t+1}\right) = \frac{j}{t(t+1)}$, and for $0 \leq j \leq t$, we have $a_{2j+1} = \left(\frac{j+1}{t+1} - \frac{j}{t}\right) = \frac{t-j}{t(t+1)}$.

Let ϕ be the labeling described above. Since k is constant, by Lemma 10, the number of bad triples is asymptotically $\sum_{i=1}^k (a_{i-1} + a_i) a_i (a_i + a_{i+1}) \binom{n}{3}$. Let A be this sum taken over even indices i , and let B be the sum taken over odd indices i . We compute

$$\begin{aligned} A &= \sum_{j=1}^t (a_{2j-1} + a_{2j}) a_{2j} (a_{2j} + a_{2j+1}) \cdot \binom{n}{3} \\ &= \binom{n}{3} \sum_{j=1}^t \frac{(t - (j - 1)) + j}{t(t + 1)} \cdot \frac{j}{t(t + 1)} \cdot \frac{j + (t - j)}{t(t + 1)} \\ &= \binom{n}{3} \frac{1}{(t(t + 1))^2} \sum_{j=1}^t j \\ &= \frac{1}{2t(t + 1)} \binom{n}{3} \end{aligned}$$

The computation for B is symmetric and also gives $B = \frac{1}{2t(t+1)} \binom{n}{3}$. Since the total number of bad triples is asymptotic to $A + B$, we have $f(n, k) \geq \left(1 - \frac{1}{t(t+1)} - o(1)\right) \binom{n}{3}$. \square

Corollary 12. *If $s \geq 2$ and s is even, then $\vec{\text{ex}}(n, P_s^{(3)}) \geq \left(1 - \frac{4}{(s-2)s} + o(1)\right) \binom{n}{3}$.*

We conjecture that these constructions are asymptotically optimal.

Conjecture 13. *If $a = \lfloor (k+1)^2/4 \rfloor$, then $f(n, k) = \left(1 - \frac{1}{a} + o(1)\right) \binom{n}{3}$. Equivalently, if $b = \lfloor (s-1)^2/4 \rfloor$, then $\vec{\text{ex}}(n, P_s^{(3)}) = \left(1 - \frac{1}{b} + o(1)\right) \binom{n}{3}$.*

Our goal in the remainder of this section is to show that if k is odd and ϕ is monotone, then ϕ has at least $\left(\frac{4}{(k+1)^2} - o(1)\right) \binom{n}{3}$ bad triples. This shows that the construction in Proposition 8 is asymptotically optimal within the class of monotone labelings. For an edge-labeling ϕ , let $\text{cost}(\phi)$ be the number of bad triples.

We believe that there exists an optimal monotone labeling of $\vec{K}_n^{(2)}$. It is at least true that when $k \geq 2$, an optimal $[k]$ -edge-labeling ϕ of $\vec{K}_n^{(2)}$ does not contain a triple uvw with $u < v < w$, $\phi(uv) = k$ and $\phi(vw) = 1$. Indeed, if there is such a triple, then we may fix v and assume that u is the minimum vertex such that $\phi(uv) = k$ and w is the maximum vertex such that $\phi(vw) = 1$. Modify ϕ to obtain ϕ' by setting $\phi'(uv) = 1$ and $\phi'(vw) = k$. We have $\text{cost}(\phi') = \text{cost}(\phi) + a - b$, where a is the number of triples that are good in ϕ but bad in ϕ' , and b is the number of triples that are bad in ϕ but good in ϕ' . We show that $b > a$, contradicting that ϕ is optimal. By maximality of w , for each w' with $w' > w$, the triple uvw' contributes to b . Similarly by minimality of u , every triple $u'vw$ with $u' < u$ also contributes to b . Also, the triple uvw itself contributes to b . Therefore $b \geq (n-w) + (u-1) + 1$. Every triple contributing to a contains uv (and therefore has the form $u'uv$ for some $u' < u$), or vw (and therefore has the form vwv' for some $v' > w$). It follows that $a \leq (u-1) + (n-w) < (n-w) + (u-1) + 1 \leq b$.

For a monotone $[k]$ -edge-labeling ϕ of $\vec{K}_n^{(2)}$ define Φ_L and Φ_R as follows. We set $\Phi_L(1) = 0$ and $\Phi_L(v) = \max\{\phi(uv) : u < v\}$ for $v > 1$. We set $\Phi_R(n) = k+1$ and $\Phi_R(v) = \min\{\phi(vw) : w > v\}$ for $v < n$. Note that by monotonicity of ϕ , if $u < v$, then $\Phi_L(u) \leq \Phi_R(u) \leq \phi(uv) \leq \Phi_L(v) \leq \Phi_R(v)$.

Using Φ_L and Φ_R , we construct two interval partitions of $V(\vec{K}_n^{(2)})$. For $0 \leq i \leq k+1$, let $X_i = \{v \in V(\vec{K}_n^{(2)}) : \Phi_L(v) = i\}$ and let $\hat{X}_i = \{v \in V(\vec{K}_n^{(2)}) : \Phi_R(v) = i\}$. Note that $X_0 = \{1\}$, $\hat{X}_0 = \emptyset$, $X_{k+1} = \emptyset$, and $\hat{X}_{k+1} = \{n\}$. Since Φ_L and Φ_R inherit the monotonicity of ϕ , both (X_0, \dots, X_{k+1}) and $(\hat{X}_0, \dots, \hat{X}_{k+1})$ are interval partitions of $V(\vec{K}_n^{(2)})$. Our next lemma shows that these partitions are very similar.

Lemma 14. *The symmetric difference of X_i and \hat{X}_i has size at most 2, and so $||X_i| - |\hat{X}_i|| \leq 2$.*

Proof. Let $u, v \in X_i$ with $u < v$. Since $i = \Phi_L(u) \leq \Phi_R(u) \leq \phi(uv) \leq \Phi_L(v) = i$ it follows that $\Phi_R(u) = i$, and so $u \in \hat{X}_i$. Therefore $X_i - \hat{X}_i \subseteq \{\max X_i\}$. Similarly, $\hat{X}_i - X_i \subseteq \{\min \hat{X}_i\}$. \square

A key step in our proof is the following bound on the sizes of the parts.

Lemma 15. *Let ϕ be an monotone $[k]$ -edge-labeling of $\vec{K}_n^{(2)}$ that minimizes the number of bad triples, and define Φ_L and (X_0, \dots, X_{k+1}) as above. For $2 \leq i \leq k$, we have $|X_i| + |X_{i+1}| \leq |X_{i-2}| + |X_{i-1}| + 2$.*

Proof. First, suppose that X_i is nonempty. Let $v = \min X_i$ and let u be the least vertex such that $\phi(uv) = i$. (Note that u exists since $\Phi_L(v) = i > 0$.) If it exists, let w be the least vertex in X_{i+1} with $\phi(vw) = i+1$. We say that an edge is *long* if its endpoints are in distinct non-consecutive parts in (X_0, \dots, X_{k+1}) .

Obtain ϕ' from ϕ by reducing by 1 the labels on vw (if w exists), uv , all long edges $u'u$ such that $u' < u$ and $\phi(u'u) = i-1$, and all long edges $v'v$ such that $v' < v$ and $\phi(v'v) = i$. Note that ϕ' is a monotone labeling. We have $\text{cost}(\phi') = \text{cost}(\phi) + a - b$, where a is the number of triples xyz with $x < y < z$ which are bad in ϕ' and good in ϕ , and b is the number of triples xyz with $x < y < z$ which are good in ϕ' and bad in ϕ . Note that if xyz is a triple with $x < y < z$ and ϕ and ϕ' agree on both xy and yz , then of course xyz contributes to neither a nor b . Also, if ϕ and ϕ' disagree on xy and yz , then also xyz does not contribute to a or b since $\phi'(xy) = \phi(xy) - 1$ and $\phi'(yz) = \phi(yz) - 1$. So each triple xyz contributing to a or b has one pair where ϕ and ϕ' agree, and one pair where ϕ and ϕ' disagree. Suppose that xyz contributes to a . It follows that $\phi'(xy) = \phi(xy)$ and $\phi'(yz) = \phi(yz) - 1 = \phi(xy)$. Note that yz is not a long edge $u'u$, since $\phi(xu') \leq \Phi_L(u') \leq \Phi_L(u) - 2 \leq i - 3$ and $\phi(u'u) = i - 1$, implying that $xu'u$ is still good in ϕ' . Similarly, yz is not a long edge $v'v$. So $yz \in \{uv, vw\}$.

If $xyz = xuv$, then $\phi(xu) = \phi'(xu) = \phi'(uv) = i - 1$ and hence $u \in X_{i-1}$. It follows that $x \in X_{i-2} \cup X_{i-1}$ since xu is not a long edge. So $xyz = xuv$ implies that $x < u$ and $x \in X_{i-2} \cup X_{i-1}$.

If $xyz = xvw$, then $\phi(xv) = \phi'(xv) = \phi'(vw) = i$. Since u is the minimum vertex with $\phi(uv) = i$, we have $u \leq x$. Moreover, $xyz \neq uvw$ since ϕ and ϕ' both disagree on uv and vw . Hence $u < x < v$. Also, $x \in X_{i-1}$ since $\phi(xv) = \phi'(xv)$, and so xv is not a long edge with label i . Combining both cases, the contributions xyz to a arise from distinct $x \in X_{i-2} \cup X_{i-1}$, and it follows that $a \leq |X_{i-2}| + |X_{i-1}|$.

It remains to show that $b \geq |X_i| + |X_{i+1}| - 2$. Let $z \in X_i \cup X_{i+1}$ such that $z > v$ and $z \neq w$ (if w exists). Note that $\phi(vz) = \phi'(vz)$, since vw is the only edge incident to the right of v where ϕ and ϕ' disagree. If $\phi(vz) = i$, then uvz contributes to b since $\phi(uv) = i$ but $\phi'(uv) = i - 1$. If $\phi(vz) > i$, then $i < \phi(vz) \leq \Phi_L(z) \leq i + 1$ and so $z \in X_{i+1}$. Hence w exists and by minimality of w we have that $w < z$. Since $\phi(vw) = i + 1$ and also $i + 1 = \Phi_L(w) \leq \phi(wz) \leq \Phi_L(z) \leq i + 1$, we have also $\phi(wz) = i + 1$. But $\phi'(vw) = i$ and $\phi'(wz) = \phi(wz) = i + 1$, and it follows that vwz contributes to b . Therefore $b \geq |X_i| + |X_{i+1}| - 2$.

Since ϕ minimizes the number of bad triples among monotone labelings, it follows that $\text{cost}(\phi') \geq \text{cost}(\phi)$, giving $a \geq b$.

Finally, we consider the case that $X_i = \emptyset$. If $X_{i+1} = \emptyset$ also, then the inequality holds trivially. Let $v = \min\{X_{i+1}\}$. Since $\Phi_L(v) = i + 1$, there is a vertex u such that $u < v$ and $\phi(uv) = i + 1$. Since $X_i = \emptyset$, it follows that $\Phi_L(u) \leq i - 1$, implying that uv is a long edge. Obtain ϕ' from ϕ by reducing $\phi(uv)$ by 1, and leaving all other labels the same. As before, we have $\text{cost}(\phi') = \text{cost}(\phi) + a - b$, where a is the number of triples xyz that are good in ϕ and bad in ϕ' , and b is the number of triples xyz that are bad in ϕ and good in ϕ' . A contribution xyz to a requires that $yz = uv$ and $\phi'(xu) = \phi'(uv) = i$, but $\phi'(xu) = \phi(xu) \leq \Phi_L(xu) \leq i - 1$, and so $a = 0$. Also, if $z > w$ and $z \in X_{i+1}$, then uvz contributes to b since $\phi'(vz) = \phi(vz) = i + 1$ and $\phi'(uv) = \phi(uv) - 1 = i$. It follows that $b \geq |X_{i+1}| - 1$ and by optimality of ϕ , we obtain $0 = a \geq b \geq |X_{i+1}| - 1$, and the inequality follows. \square

Corollary 16. *Let ϕ be a monotone $[k]$ -edge-labeling of $\vec{K}_n^{(2)}$ that minimizes the number of bad triples, and define Φ_R and $(\hat{X}_0, \dots, \hat{X}_{k+1})$ as above. For $1 \leq j \leq k - 1$, we have $|\hat{X}_{j-1}| + |\hat{X}_j| \leq |\hat{X}_{j+1}| + |\hat{X}_{j+2}| + 2$.*

Proof. Obtain ϕ' from ϕ by reversing the order of vertices in $\vec{K}_n^{(2)}$ and inverting the labels, so that $\phi'(uv) = (k + 1) - \phi(v^*u^*)$, where $v^* = (n + 1) - v$ and $u^* = (n + 1) - u$. Note that ϕ' is a monotone $[k]$ -edge-labeling with $\text{cost}(\phi') = \text{cost}(\phi)$. Moreover, defining Φ'_L with respect to ϕ' and the corresponding partition (X'_0, \dots, X'_{k+1}) , we have that $\Phi'_L(u) = (k + 1) - \Phi_R(u^*)$ where $u^* = (n + 1) - u$ and $|X'_i| = |\hat{X}_{(k+1)-i}|$. Applying Lemma 15 to ϕ' with $i = (k + 1) - j$ gives the result. \square

Theorem 17. *Let ϕ be a monotone $[k]$ -edge-labeling of $\vec{K}_n^{(2)}$ that minimizes the number of bad triples. If k is odd, then $\text{cost}(\phi) \geq (1 - o(1)) \frac{4}{(k+1)^2} \binom{n}{3}$.*

Proof. Define $\Phi_L, \Phi_R, (X_0, \dots, X_{k+1})$, and $(\hat{X}_0, \dots, \hat{X}_{k+1})$ as above, and let $m = (k - 1)/2$. For $0 \leq \ell \leq m$, let $a_\ell = |X_{2\ell}| + |X_{2\ell+1}|$ and let $\hat{a}_\ell = |\hat{X}_{2\ell}| + |\hat{X}_{2\ell+1}|$. By Lemma 15 with $i = 2\ell$, we have $a_{\ell-1} \geq a_\ell - 2$ for $1 \leq \ell \leq m$. It follows that $a_0 \geq a_1 - 2 \geq \dots \geq a_m - 2m = a_m - (k - 1)$. With two applications of Lemma 14, we have $a_0 \leq \hat{a}_0 + 4$. By Corollary 16 with $j = 2\ell - 1$, we have $\hat{a}_{\ell-1} \leq \hat{a}_\ell + 2$ for $1 \leq \ell \leq m$, and it follows that $\hat{a}_0 \leq \hat{a}_1 + 2 \leq \dots \leq \hat{a}_m + 2m = \hat{a}_m + (k - 1)$. Chaining the inequalities gives $-(k - 1) + a_m \leq \dots \leq a_0 \leq \hat{a}_0 + 4 \leq \dots \leq \hat{a}_m + k + 3 \leq a_m + k + 7$. It follows that $-2\ell + a_\ell$ is in the interval $[a_m - (k - 1), a_m + (k + 7)]$, and so a_ℓ is in the interval $[a_m - (k - 1), a_m + (k + 7) + 2\ell]$. Since $2\ell \leq k - 1$, we have $|a_\ell - a_m| \leq 2k + 6$ and so each pair in $\{a_0, \dots, a_m\}$ differs by at most $4k + 12$. Since $\sum_{\ell=0}^m a_\ell = n$, it follows that $a_\ell = n/(m + 1) + O(k)$.

Similarly, for $0 \leq \ell \leq m$, let $b_\ell = |X_{2\ell+1}| + |X_{2\ell+2}|$ and $\hat{b}_\ell = |\hat{X}_{2\ell+1}| + |\hat{X}_{2\ell+2}|$. By Lemma 15 with $i = 2\ell + 1$, we have $b_{\ell-1} \geq b_\ell - 2$ for $1 \leq \ell \leq m$. It follows that $b_0 \geq b_1 - 2 \geq \dots \geq b_m - (k - 1)$. Similarly, $b_0 \leq \hat{b}_0 + 4$. Also, Corollary 16 with $j = 2\ell$ gives $\hat{b}_{\ell-1} \leq \hat{b}_\ell + 2$ for $1 \leq \ell \leq m$. Therefore $\hat{b}_0 \leq \hat{b}_1 + 2 \leq \dots \leq \hat{b}_m + (k - 1)$. Combining the inequalities gives $-(k - 1) + b_m \leq \dots \leq b_0 \leq \hat{b}_0 + 4 \leq \hat{b}_m + (k + 3) \leq b_m + (k + 7)$. As above, b_ℓ and $b_{\ell'}$ differ by at most $4k + 12$. Again, $\sum_{\ell=0}^m b_\ell = n$, and so $b_\ell = n/(m + 1) + O(k)$.

Let $0 \leq \ell \leq m$. We claim that $X_{2\ell}$ and $X_{2\ell+2}$ differ in size by at most $O(k)$. Indeed, both a_ℓ and b_ℓ equal $n/(m + 1) + O(k)$, and so $|a_\ell - b_\ell| \leq O(k)$. Since $a_\ell = |X_{2\ell}| + |X_{2\ell+1}|$ and $b_\ell = |X_{2\ell+1}| + |X_{2\ell+2}|$, we have

that $a_\ell - b_\ell = |X_{2\ell}| - |X_{2\ell+2}|$ and the claim follows. Therefore each pair of parts in $\{X_0, X_2, X_4, \dots, X_{k+1}\}$ differs in size by at most $O(k^2)$. Since $|X_{k+1}| = 0$, it follows that each part with even index has size at most $O(k^2)$. Since $a_\ell = |X_{2\ell}| + |X_{2\ell+1}| = n/(m+1) + O(k)$, it follows that each part with odd index has size $n/(m+1) - O(k^2)$. Since each triple uvw with $u, v, w \in X_i$ satisfies $\phi(uv) = \phi(vw) = i$, the number of bad triples in ϕ is at least $\sum_{\ell=0}^m \binom{|X_{2\ell+1}|}{3}$, and $\sum_{\ell=0}^m \binom{|X_{2\ell+1}|}{3} \geq (m+1) \binom{(n/(m+1)) - O(k^2)}{3} = (1 - o(1)) \frac{1}{(m+1)^2} \binom{n}{3}$. \square

Let k be even, let ϕ be an optimal monotone $[k]$ -edge-labeling of $\vec{K}_n^{(2)}$, and define the parts (X_0, \dots, X_{k+1}) as above. Similar arguments as in Theorem 17 can be used to obtain the sizes of the parts asymptotically, and these match the sizes of the corresponding parts in our construction in Lemma 11. However, bounding the number of bad triples in ϕ for even k is more complicated since consecutive parts X_i and X_{i+1} are both linear in n , making the triples with two vertices in one of $\{X_i, X_{i+1}\}$ and one in the other significant.

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