Turán Numbers of Ordered Tight Hyperpaths

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Abstract

An ordered hypergraph is a hypergraph $G$ whose vertex set $V(G)$ is linearly ordered. We find the Turán numbers for the $r$-uniform $s$-vertex tight path $P_s^{(r)}$ (with vertices in the natural order) exactly when $r \leq s < 2r$ and $n$ is even; our results imply $\varepsilon(n, P_s^{(r)}) = (1 - \frac{1}{2^{r-1}} + o(1))\binom{n}{r}$ when $r \leq s < 2r$. When $r \geq 2s$, the asymptotics of $\varepsilon(n, P_s^{(r)})$ remain open. For $r = 3$, we give a construction of an $r$-uniform $n$-vertex hypergraph not containing $P_s^{(r)}$ which we conjecture to be asymptotically extremal.

1 Introduction

The Turán number of an $r$-uniform hypergraph $H$, denoted $\varepsilon(n, H)$, is the maximum number of edges in an $r$-uniform $n$-vertex graph $G$ that does not contain $H$ as a subgraph. Bounding Turán numbers is a classical problem in extremal graph theory. The best known general bounds on the Turán numbers of the $r$-uniform $s$-vertex complete hypergraph $K_s^{(r)}$ are

$$(1 - \frac{1}{r-1})\binom{s}{r-1} - o(1)\binom{n}{r} \leq \varepsilon(n, K_s^{(r)}) \leq (1 - \frac{1}{r-1})\binom{s-1}{r-1} + o(1)\binom{n}{r},$$

with lower bound due to Sidorenko [9] and upper bound due to de Caen [1].

An ordered hypergraph is a hypergraph $G$ whose vertices are linearly ordered. For an ordered hypergraph $G$, the underlying hypergraph is the ordinary hypergraph obtained from $G$ by discarding the order on $V(G)$. For vertices $u$ and $v$ in an ordered hypergraph $G$, we write $u <_G v$, or $u < v$ when $G$ is clear from context, if $u$ appears before $v$ in the ordering of $V(G)$. If $G$ and $H$ are ordered hypergraphs, then $G$ is a subgraph of $H$, denoted $G \subseteq H$, if there is an injection $f : V(G) \to V(H)$ such that $u <_G v$ if and only if $f(u) <_H f(v)$ and $e \in E(G)$ implies $f(e) \in E(H)$, where $f(e) = \{f(v) : v \in e\}$. When $H$ is an $r$-uniform ordered hypergraph, we use $\varepsilon(n, H)$ to denote the analogous ordered Turán number, so that $\varepsilon(n, H)$ is the maximum number of edges in an $r$-uniform $n$-vertex ordered hypergraph not containing $H$ as a subgraph.

For graphs, ordered Turán numbers behave somewhat analogously to ordinary Turán numbers. The interval chromatic number of an ordered graph $G$, denoted $\chi_i(G)$, is the minimum $k$ such that $V(G)$ can be partitioned into $k$ intervals, each of which is an independent set. Although computing the chromatic number of an ordinary graph is NP-hard, an easy greedy algorithm computes $\chi_i(G)$ for an ordered graph $G$. Pach and Tardos [8] obtained an ordered analogue of the Erdős–Stone Theorem, showing that for each ordered graph $H$, we have $\varepsilon(n, H) = (1 - \frac{1}{\chi_i(H)-1} + o(1))\binom{n}{\chi_i(H)}$. Like the Erdős–Stone Theorem, this establishes the Turán numbers asymptotically for each ordered graph $G$ with $\chi_i(G) > 2$. It is therefore natural to focus on ordered graphs $G$ with $\chi_i(G) = 2$ and ordered hypergraphs.

A graph $G$ is a forest if $G$ has no cycles. Using classical Turán Theory, it is straightforward to show that $\varepsilon(n, G) \geq \Omega(n^{1+\varepsilon})$ for some positive $\varepsilon$ unless $G$ is an ordered forest with $\chi_i(G) = 2$. Pach and Tardos [8] conjectured that if $G$ is an ordered forest with $\chi_i(G) = 2$, then $\varepsilon(n, G) \leq n(\log n)^{O(1)}$. Korándi, Tardos, Tomon, and Weidert [7] made progress on the conjecture by proving that $\varepsilon(n, G) \leq n^{1+o(1)}$ when $G$ is an ordered forest with $\chi_i(G) = 2$. For a family of ordered graphs $\mathcal{G}$, we define $\varepsilon(n, \mathcal{G})$ to be the maximum number of edges in an $n$-vertex ordered graph that contains no member of $\mathcal{G}$ as a subgraph. A bordered cycle is an ordered graph $G$ whose underlying graph is a cycle, whose ordering has intervals $X$ and $Y$ with $X < Y$ such that each edge in $G$ has an endpoint in $X$ and an endpoint in $Y$ (implying $\chi_i(G) \leq 2$), and contains the edge joining $\min X$ and $\max Y$ and the edge joining $\max X$ and $\min Y$. Győri, Korándi, Methuku, Tomon,
Tomkincs, and Vizer [6] proved that $\text{ex}(n, G_k) = \Theta(n^{1+1/k})$, where $G_k$ is the family of bounded cycles on at most $2k$ vertices.

The $r$-uniform $s$-vertex \textit{natural path}, denoted $P_s(r)$, has vertex set $\{v_1, \ldots, v_s\}$ in the natural order $v_1 < \cdots < v_s$ with $E(P_s(r))$ consisting of all intervals of size $r$. The underlying hypergraph of $P_s(r)$ is the well-known \textit{tight path} $P_s(r)$. The length of a path is the number of edges in the path, and so both $P_s(r)$ and $P_s(r)$ have length $s - r + 1$. A special case of a conjecture by Kalai [2] states that for $n \geq r \geq 2$ and $s \geq r$, we have $\text{ex}(n, P_s(r)) \leq \binom{n}{r}$, which remains open. F"uredi, Jiang, Kostochka, Mubayi, and Verstra"ete [3] proved that $\text{ex}(n, P_s^3(n)) = \binom{n}{2}$ for $n \geq 5$. In a later paper [4], the same authors proved that if $s \geq r$, then $\text{ex}(n, P_s^3(n)) \leq \frac{1}{2} \left( s - r + 1 + \left\lceil \frac{4}{(s-1)r} \right\rceil \right) \binom{n}{r}$ when $r$ is odd. Few results on Turán numbers of ordered hypergraphs are known. In classical Turán theory, an $r$-uniform hypergraph $G$ satisfies $\text{ex}(n, G) = o(n^r)$ if and only if $G$ is \textit{r-partite}, meaning that there is a partition of $V(G)$ into $r$ parts such that each edge in $G$ has one vertex in each part. The analogous statement holds for ordered hypergraphs: an $r$-uniform hypergraph $G$ satisfies $\text{ex}(n, G) = o(n^r)$ if and only if $G$ is \textit{r-interval-partite}, meaning that $V(G)$ can be partitioned into $r$ intervals such that each edge in $G$ has one vertex in each interval.

For $s > r$, the natural paths $P_s(r)$ are not $r$-interval-partite, and so $\text{ex}(n, P_s^3(n)) \geq \Omega(n^r)$. The vertices of a tight path can be arranged in a different order to give an ordered $r$-interval-partite hypergraph. The $r$-uniform $s$-vertex \textit{crossing path}, is a tight path whose vertices are ordered as follows. Arrange the $s$ vertices in an $r \times n$ grid with $r$ rows $R_1, \ldots, R_r$ and $s$ columns such that any empty cells form a suffix of the last column. Let $t = \lfloor s/r \rfloor$, and let $C_1, \ldots, C_t$ be the columns of the grid. The ordering on the vertices of $Q_s^r(n)$ satisfies $R_1 < \cdots < R_r$, where the vertices in each $R_i$ are ordered from $C_1$ to $C_t$ (or $C_{t-1}$ if $R_i$ has no vertex in row $C_t$). The edges of $Q_s^r(n)$ are the intervals of size $r$ in the alternative vertex ordering such that $C_1 < \cdots < C_t$, where the vertices in each $C_j$ are ordered from $R_1$ to $R_r$ (or, in the case of $C_t$, from $R_1$ to the last occupied row). Since each edge in $Q_s^r(n)$ has one vertex in each row and $R_1 < \cdots < R_r$, it follows that $Q_s^r(n)$ is $r$-interval-partite. F"uredi, Jiang, Kostochka, Mubayi, and Verstra"ete [5] proved that $\text{ex}(n, Q_s^r(n)) = \binom{n}{r} - \binom{n-(s-r)}{r}$ when $r \leq s \leq 2r$ and is $\Theta(n^{r-1}\log n)$ when $s > 2r$. Since $\binom{n}{r} - \binom{n-(s-r)}{r} = (1 + o(1))n(s-r)n^{r-1}$, it follows that always $\text{ex}(n, Q_s^r(n)) = O(n^{r-1}\log(n))$. A hypergraph $F$ is a forest if the edges of $F$ can be ordered as $e_1, \ldots, e_m$ such that for each $i$, the edge $e_i$ is the union of a subset of an earlier edge $e_j$ with $j < i$ and vertices that are not contained in any edge in $\{e_1, \ldots, e_{i-1}\}$. In classical Turán theory, we have $\text{ex}(n, F) \leq O(n^{r-1})$ for each $r$-uniform forest $F$. Generalizing the Pach-Tardos conjecture, F"uredi et al [5] conjectured that $\text{ex}(n, F) = O(n^{r-1}\cdot \text{polylog}(n))$ when $F$ is an $r$-uniform $r$-interval-partite forest.

We are interested in $\text{ex}(n, P_s^3(n))$. In Section 2, we obtain $\overline{\text{ex}}(n, P_s^3(n))$ exactly when $s \leq 2r-1$ and $n$ is even, implying that $\overline{\text{ex}}(n, P_s^3(n)) = (1 - \frac{4}{s-1}) + o(1)) \binom{n}{r}$ when $r \leq s \leq 2r-1$. When $s \geq 2r$, determining the asymptotics of $\overline{\text{ex}}(n, P_s^3(n))$ remains open. When $r$ divides $s$, our fractional results in Section 3 imply $\text{ex}(n, P_s^3(n)) \leq (1 - (\frac{s}{r}) + o(1)) \binom{n}{r}$. When $r - s$ divides $s - 1$, partitioning an interval of $s$ vertices into $(s-1)/(r-1)$ parts of equal size and removing edges with all vertices in a single part shows that $\text{ex}(n, P_s^3(n)) \geq (1 - \frac{4}{s-1}) - o(1)) \binom{n}{r}$, matching the Szidorenko lower bound on $\text{ex}(n, K_s^3(n))$ even though $\text{ex}(n, P_s^3(n)) \leq \text{ex}(n, K_s^3(n)) = \text{ex}(n, K_s^3(n))$. When $r$ and $s$ do not have convenient divisibility relationships, obtaining bounds on $\text{ex}(n, P_s^3(n))$ may involve additional subtleties. Szidorenko’s lower bound on $\text{ex}(n, K_s^3(n))$ holds for general $r$ and $s$; in fact, the argument shows that $\text{ex}(n, C_s^3(n)) \geq (1 - (\frac{4}{s-1}) - o(1)) \binom{n}{r}$, which is an $r$-uniform $s$-vertex ordered tight cycle, with vertex set $\{v_0, \ldots, v_{s-1}\}$ in the natural order and edge set $\{e_0, \ldots, e_{s-1}\}$, where $e_j = \{v_j, \ldots, v_{j+s-1}\}$ (subscript arithmetic modulo $s$).

We study the case $s = 3$ in Section 4. When $s$ is odd, we have that $r - s$ divides $s - 1$ and the same construction as above gives $\text{ex}(n, P_s^3(n)) \geq (1 - \frac{4}{s-1}) - o(1)) \binom{n}{r}$. When $s$ is even, we give a construction that improves the lower bound to $\text{ex}(n, P_s^3(n)) \geq (1 - \frac{4}{r(s-2)} - o(1)) \binom{n}{r}$. We conjecture that these bounds are asymptotically sharp. An ordered hypergraph $G$ is \textit{monotone} if, for each edge $\{u, v, w\} \in E(G)$ with $u < v < w$, we have $\ell(uv) \leq \ell(uw) \leq \ell(vw)$, where $\ell(xy)$ is the length of a longest ordered tight path whose last two
vertices are $x$ followed by $y$ (see Section 4 for an equivalent, but perhaps more natural, formulation). As some partial evidence for this conjecture, we show that if $s$ is odd and $G$ is a monotone $n$-vertex ordered hypergraph not containing $P_6^{(s)}$, then $|E(G)| \leq (1 - \frac{1}{(s-1)!}) \binom{n}{3}$. The first unresolved case is that of $P_6^{(3)}$, with best known bounds $(\frac{4}{9} - o(1)) \binom{n}{3} \leq \bar{e}(n, P_6^{(3)}) \leq (\frac{5}{9} + o(1)) \binom{n}{3}.

2 Exact Results for Short Paths

In this section, our aim is to establish $\bar{e}(n, P_s^{(r)})$ exactly when $r \leq s \leq 2r - 1$ and $n$ is even. If $G \subseteq \tilde{K}_n^{(r)}$ and $G$ does not contain $H$, then each copy of $H$ in $\tilde{K}_n^{(r)}$ has some edge in $\tilde{G}$. An $H$-transversal in $\tilde{K}_n^{(r)}$ is a graph $G' \subseteq \tilde{K}_n^{(r)}$ such that every copy of $H$ in $\tilde{K}_n^{(r)}$ has at least one edge in $G'$. The transversal number of $H$, denoted $\bar{\tau}(n, H)$, is the minimum number of edges in an $H$-transversal. Note that $e(n, H) + \bar{\tau}(n, H) = |E(\tilde{K}_n^{(r)})| = \binom{n}{r}$.

We use $[n]$ for the vertex set of $\tilde{K}_n^{(r)}$. For vertex sets $A$ and $B$ in an ordered graph $G$, we write $A < B$ if $a < b$ for all $a \in A$ and $b \in B$. The reflection of a vertex $u$ in $\tilde{K}_n^{(r)}$ is the vertex $n + 1 - u$. An interval partition of $\tilde{K}_n^{(r)}$ is a list of disjoint vertex subsets $(X_1, \ldots, X_k)$ whose union is $[n]$ such that each $X_i$ is an interval in $[n]$ and $X_i < X_j$ when $i < j$. A set of vertices $S$ is $m$-left-biased if $\tilde{K}_n^{(r)}$ has an interval partition $(X, Y, Z)$ such that $|X| = |Z|$, $|X \cap S| = m$, and $|Z \cap S| = 0$. Similarly, $S$ is $m$-right-biased if $\tilde{K}_n^{(r)}$ has an interval partition $(X, Y, Z)$ such that $|X| = |Z|$, $|X \cap S| = 0$, and $|Z \cap S| = m$. We say that $S$ is $m$-biased if $S$ is $m$-left-biased or $m$-right-biased. Let $h(n, t, m)$ be the number of $t$-sets that are $m$-left-biased in $\tilde{K}_n^{(r)}$. Note that for even $n$, summing the $m$-left-biased $t$-sets whose $m$th vertex is at index $k$ shows that $h(n, t, m) = \sum_{k=m}^{n/2} \binom{n-k+1}{m-1} \binom{n-k}{t-m}$ when $1 \leq m \leq t$.

Our next theorem gives a lower bound on $\bar{e}(n, P_s^{(r)})$ by constructing an $\tilde{P}_s^{(r)}$-transversal when $r \leq s \leq 2r - 1$. In fact, we construct a $LP_s^{(r)}$-transversal, where $LP_s^{(r)}$ is the loose path obtained from $\tilde{P}_s^{(r)}$ by removing all but the first and last edges. Note that the two edges in $LP_s^{(r)}$ intersect when $r < s \leq 2r - 1$.

**Theorem 1.** Let $n$ be even, let $r \leq s \leq 2r - 1$, and let $m = |E(\tilde{P}_s^{(r)})| = s - r + 1$. We have $\bar{\tau}(n, \tilde{P}_s^{(r)}) \leq 2h(n, r, m) + h(n, r - 1, m)$.

**Proof.** The condition $r \leq s \leq 2r - 1$ translates to $1 \leq m \leq r$. Let $G$ be the subgraph of $\tilde{K}_n^{(r)}$ such that $E(G) = E_1 \cup E_2$, where $E_1$ is the family of $r$-sets that are $m$-biased and $E_2$ is the family of $r$-sets whose $m$th and last vertices are reflections of one another. Since $m > 0$, the $m$-biased $r$-sets are the disjoint union of the $m$-left-biased $r$-sets and the $m$-right-biased $r$-sets, both of which have size $h(n, r, m)$, and so $|E_1| = 2h(n, r, m)$. Also, when $m < r$, removing the last vertex from an $r$-set whose $m$th and last points are reflections of one another gives an $(r - 1)$-set that is $m$-left-biased and conversely, and so $|E_2| = h(n, r - 1, m)$.

(No third when $r = m$, we have $h(n, r - 1, m) = 0$.)

It remains to show that every copy of $LP_s^{(r)}$ in $\tilde{K}_n^{(r)}$ has an edge in $G$. Let $Q$ be such a copy, and let $(X, Y, Z)$ be the interval partition of $\tilde{K}_n^{(r)}$ that minimizes $|X|$ subject to $|X| = |Z|$ and $\max(|X \cap V(Q)|, |Z \cap V(Q)|) \geq m$. Such a partition exists, or else $Q$ has at most $m - 1$ vertices in both the left and right halves of $\tilde{K}_n^{(r)}$, which would imply $s = |V(Q)| \leq 2(m - 1) = 2(s - r)$, contradicting $s \leq 2r - 1$. Let $u = |X|$.

Suppose first that $|X \cap V(Q)| = |Z \cap V(Q)| = m$. In this case, it must be that both $u$ and its reflection $n + 1 - u$ are vertices in $Q$. Note that $s = r + (m - 1)$. Deleting the last $m - 1$ vertices in $Q$ gives the first edge $e \in E(Q)$ whose $m$th vertex is $u$ and whose last vertex is $n + 1 - u$, implying that $e \in E_2 \subseteq E(G)$.

Otherwise, one of $|X \cap V(Q)|, |Z \cap V(Q)|$ equals $m$ and the other is at most $m - 1$. We show that $Q$ has an edge in $E_1$. Suppose that $|X \cap V(Q)| = m$ and $|Z \cap V(Q)| \leq m - 1$. Let $e$ be the first edge in $Q$ (which is obtained by deleting the last $m - 1$ vertices of $Q$). Since $s \geq m + (m - 1)$, none of the deleted vertices are in $X$. It follows that $|X \cap e| = m$ and $|Z \cap e| = 0$. So $e$ is $m$-left-biased, and therefore $e \in E_1 \subseteq E(G)$.

If instead $|X \cap V(Q)| \leq m - 1$ and $|Z \cap V(Q)| = m$, then a similar argument shows that the last edge $e$ in $Q$ is $m$-right-biased, also implying $e \in E_1 \subseteq E(G)$.
Our next theorem obtains a large family of edge-disjoint copies of \( \tilde{P}_s^{(r)} \). For an \( r \)-uniform ordered hypergraph \( H \), the \( H \)-packing number, denoted \( \tilde{\nu}(n, H) \), is the maximum size of an edge-disjoint family of copies of \( H \) in \( K_n^r \). Clearly, \( \tilde{\nu}(n, H) \leq \tilde{\tau}(n, H) \).

**Theorem 2.** Let \( n \) be even, let \( r \leq s \leq 2r-1 \), and let \( m = \left| E(\tilde{P}_s^{(r)}) \right| = s - r + 1 \). We have \( \tilde{\nu}(n, \tilde{P}_s^{(r)}) \geq 2h(n, r, m) + h(n, r-1, m) \).

**Proof.** As in Theorem 1, let \( G \) be the subgraph of \( \tilde{K}_n^r \) with \( E(G) = E_1 \cup E_2 \), where \( E_1 \) is the family of \( r \)-sets that are \( m \)-biased and \( E_2 \) is the family of \( r \)-sets whose \( m \)th vertex and last vertex are reflections of one another. For each \( e \in E(G) \), we construct a copy \( Q_e \) of \( \tilde{P}_s^{(r)} \) such that the family of paths \( \{ Q_e : e \in E(G) \} \) is edge-disjoint.

Let \( e \in E(G) \). We construct \( Q_e \) as follows. Note that every edge in \( G \) is \( (m-1) \)-biased. The canonical interval partition of \( e \) is the interval partition \( (X, Y, Z) \) of \( \tilde{K}_n^r \) that maximizes \( |X| \) subject to \( |X| = |Z| \), \( \max\{|X \cap e|, |Z \cap e|\} = m-1 \), and \( \min\{|X \cap e|, |Z \cap e|\} = 0 \). Note that by maximality of \( |X| \), it follows that either \( \min Y \) or \( \max Y \) is a vertex in \( e \), and so the canonical interval partition of \( e \) is also the interval partition \( (X, Y, Z) \) of \( \tilde{K}_n^r \) that minimizes \( |Y| \) subject to \( |X| = |Z| \) and \( |Y \cap e| = r - (m - 1) = 2r - s \). The translation of a set of vertices \( S \subseteq V(\tilde{K}_n^r) \) by a constant \( c \), denoted \( S + c \), is the set \( \{ u + c : u \in S \} \). If \( |X \cap e| = m-1 \), then we take \( V(Q_e) \) to be the union of \( e \) and the translation \( (X \cap e) + (n - |Z|) \). Otherwise, if \( |Z \cap e| = m-1 \), then we take \( V(Q_e) \) to be the union of \( e \) and the translation \( (Z \cap e) + (n - |X|) \). In both cases, \( |V(Q_e)| = r + m - 1 = s \), and so \( Q_e \) is a copy of \( \tilde{P}_s^{(r)} \).

The core of \( Q_e \) is the set of 2r - s vertices in \( V(Q_e) \) that belong to every edge in \( Q_e \). Since \( 2r - s \geq 1 \), the core of \( Q_e \) is non-empty. Also, since \( 2r - s = s - 2(m-1) \) and \( |X \cap V(Q_e)| = 2 \), it follows that the core of \( Q_e \) equals \( Y \cap V(Q_e) \).

Because the core of \( Q_e \) consists of the 2r - s vertices in \( Q_e \) that are closest to the center of \( V(\tilde{K}_n^r) \), given any edge in \( Q_e \) we can identify the core of \( Q_e \). Moreover, since the canonical interval partition of \( e \) is the partition \( (X, Y, Z) \) of \( \tilde{K}_n^r \) given by minimizing \( |Y| \) subject to \( |X| = |Z| \) and \( |Y \cap e| = 2r - s \), each edge in \( Q_e \) also determines the canonical interval partition of \( e \).

We show that given an edge \( f \) in some path \( Q_e \) in the family \( \{ Q_e : e \in E(G) \} \), we can determine the edge \( e \in E(G) \) that generates \( Q_e \). It follows that the family is edge-disjoint. Let \( f \) be an edge in one of the paths in our collection, and recall that \( f \) determines the canonical interval partition \( (X, Y, Z) \) of the generating edge \( e \). It follows that \( V(Q_e) \) is the union of \( f \) and the translations \( (X \cap f) + (n - |Z|) \) and \( (Z \cap f) + (n - |X|) \). Note that the edge \( e \in E(G) \) that generates \( Q_e \) must be the first or last edge in \( Q_e \). We identify \( e \) as follows. If \( \min Y \) and \( \max Y \) are in \( V(Q_e) \), then \( e \in E_2 \) and \( e \) is the first edge in \( Q_e \). If \( \min Y \) is in \( V(Q_e) \) but \( \max Y \) is not, then \( e \in E_1 \) and \( e \) is also the first edge in \( Q_e \). Otherwise, \( \max Y \) is in \( V(Q_e) \) and \( \min Y \) is not, in which case \( e \) is the last edge in \( Q_e \).

The theorems give exact results on \( \tilde{\nu}(n, \tilde{P}_s^{(r)}), \tilde{\tau}(n, \tilde{P}_s^{(r)}), \) and \( \tilde{\nu}(n, \tilde{P}_s^{(r)}) \) when \( r \leq s \leq 2r-1 \).

**Corollary 3.** Let \( n \) be even, let \( r \leq s \leq 2r-1 \), and let \( m = \left| E(\tilde{P}_s^{(r)}) \right| = s - r + 1 \). We have that each parameter in \( \{ \tilde{\nu}(n, \tilde{P}_s^{(r)}), \tilde{\nu}(n, L\tilde{P}_s^{(r)}), \tilde{\tau}(n, \tilde{P}_s^{(r)}), \tilde{\tau}(n, L\tilde{P}_s^{(r)}) \} \) equals \( 2h(n, r, m) + h(n, r-1, m) \), and \( 2h(n, r, m) + h(n, r-1, m) = \frac{1}{2^{n-r}} \binom{n}{r} + O(n^{r-1}) \). Therefore \( \tilde{\nu}(n, L\tilde{P}_s^{(r)}) = \tilde{\nu}(n, \tilde{P}_s^{(r)}) = \binom{n}{r} - 2h(n, r, m) - h(n, r-1, m) = \left(1 - \frac{1}{2^{n-r}}\right) \binom{n}{r} + O(n^{r-1}) \).
and so $2^{r-m\left(\frac{n}{r}\right)}$ sets in $B$ are $m$-left-biased. Let $C$ be the family of $m$-left-biased $r$-sets. We compute $h(n, r, m) = |C| = |C \cap A| + |C \cap B| = O(n^{r-1}) + 2^{r-m\left(\frac{n}{r}\right)} = O(n^{r-1}) + \frac{1}{2^m} \binom{n}{r}$. \hfill \□

3 Fractional Variants

The transversal and packing numbers from Section 2 have fractional variants. For an ordered hypergraph $H$, a fractional transversal of the copies of $H$ in $\tilde{K}_n^{(r)}$ is a function $w$ that assigns non-negative weights to each edge in $\tilde{K}_n^{(r)}$ such that $\sum_{e \in E(H')} w(e) \geq 1$ for each copy $H'$ of $H$ in $\tilde{K}_n^{(r)}$. The fractional transversal number of $H$, denoted $\tilde{\tau}(n, H)$, is the infimum, over all fractional transversals $w$, of the sum of $w(e)$ over all edges $e \in E(\tilde{K}_n^{(r)})$. Standard compactness arguments show that the infimum is always achieved, and so we may replace infimum by minimum in the definition. Also, if $G' \subseteq \tilde{K}_n^{(r)}$ and $G'$ is an $H$-transversal, then the weight function $w$ with $w(e) = 1$ for $e \in E(G')$ and $w(e) = 0$ for $e \notin E(G')$ is a fractional transversal, and therefore $\tilde{\tau}(n, H) \leq \tilde{\tau}(n, H)$. Hence

$$\tilde{\nu}(n, H) \leq \tilde{\nu}(n, H) = \tilde{\tau}(n, H) \leq \tilde{\tau}(n, H).$$

In this section, we show that $\tilde{\nu}(n, \tilde{P}_s^{(r)})$, $\tilde{\nu}(n, \tilde{P}_s^{(r)})$, and $\tilde{\tau}(n, \tilde{P}_s^{(r)})$ are all asymptotically $\left(\left(\frac{n}{r}\right)^r + o(1)\right) \binom{n}{r}$ when $r$ divides $s$.

**Proposition 4.** If $r$ divides $s$, then $\tilde{\nu}(n, \tilde{P}_s^{(r)}) \geq \left(\left(\frac{n}{r}\right)^r + o(1)\right) \binom{n}{r}$.

**Proof.** We give a fractional $\tilde{P}_s^{(r)}$-packing of the required size. Let $k = s/r$, and without loss of generality assume that $k$ divides $n$. Let $X_1, \ldots, X_k$ be an interval partition of $\tilde{K}_n^{(r)}$ into parts of equal size. For each edge $e \in E(\tilde{K}_n^{(r)})$ with $e \subseteq X_1$, let $P_e$ be the $s$-vertex path with vertex set $\bigcup_{i=1}^{k-1} \{e + j|X_1|\}$. Note that given any edge $e' \in E(P_e)$, it is easy to recover $e$, and it follows that $\bigcup_{e \subseteq X_1} \{P_e\}$ is an edge-disjoint collection of copies of $\tilde{P}_s^{(r)}$. Therefore $\tilde{\nu}(n, \tilde{P}_s^{(r)}) \geq \binom{|X_1|}{r} \left(\frac{n}{k}\right)^r = \left(\frac{1}{k^r} + o(1)\right) \binom{n}{r}$. \hfill \□

**Proposition 5.** If $r$ divides $s$, then $\tilde{\tau}(n, \tilde{P}_s^{(r)}) \leq \left(\left(\frac{n}{r}\right)^r + o(1)\right) \binom{n}{r}$.

**Proof.** We give a fractional $\tilde{P}_s^{(r)}$-transversal. Let $k = s/r$. We may assume without loss of generality that $k$ divides $n$. Let $X_1, \ldots, X_k$ be an interval partition of $\tilde{K}_n^{(r)}$ into parts of equal size. Let $w$ be the weight function with $w(e) = r/s$ if $e$ is contained in a part in $(X_1, \ldots, X_k)$ and $w(e) = 0$ otherwise. Let $P$ be a copy of $\tilde{P}_s^{(r)}$ in $\tilde{K}_n^{(r)}$. Note that at most $r - 1$ vertices in $V(P) \cap X_i$ begin an edge with weight zero, and $P$ has at most $k(r - 1)$ such vertices. Therefore at least $s - k(r - 1)$ vertices in $P$ begin an edge with positive weight. So the edges of $P$ have total weight at least $(r/s)(s - k(r - 1))$, and this equals $w$. It follows that $w$ is a fractional $\tilde{P}_s^{(r)}$-transversal, and so $\tilde{\tau}(n, \tilde{P}_s^{(r)}) \leq \frac{1}{k^r} \cdot \binom{|X_1|}{r} = \left(\frac{n}{k}\right)^r = \left(\frac{1}{k^r} + o(1)\right) \binom{n}{r}$. \hfill \□

If $r$ divides $s$ and the integer $s/r$ also divides $n$, then Proposition 4 and Proposition 5 imply $\tilde{\nu}(n, \tilde{P}_s^{(r)}) = \tilde{\nu}(n, \tilde{P}_s^{(r)}) = \tilde{\nu}(n, \tilde{P}_s^{(r)}) = \left(\frac{n}{k}\right)^r$, where $k = s/r$.

**Theorem 6.** If $r$ divides $s$, then $\tilde{\nu}(n, \tilde{P}_s^{(r)})$, $\tilde{\nu}(n, \tilde{P}_s^{(r)})$, and $\tilde{\tau}(n, \tilde{P}_s^{(r)})$ are all asymptotically equal to $\left(\left(\frac{n}{r}\right)^r + o(1)\right) \binom{n}{r}$.\hfill 

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The smallest path to which our argument in Section 2 does not apply is $P_{2r}^r$, when $s = 2r$. In this case, our fractional results imply $\tilde{\tau}(n, P_{2r}^r) = \left(\frac{1}{2r} + o(1)\right)(n^r)$, but, at least in the case $r = 3$, we believe that $\tilde{\tau}(n, P_{2r}^r) \gg \tau(n, P_{2r}^r)$. In particular, $\tilde{\tau}(n, P_6^3) = \left(\frac{4}{3} + o(1)\right)(n^r)$ but we conjecture $\tilde{\tau}(n, P_6^3) = \left(\frac{4}{3} + o(1)\right)(n^r)$.

4 The case $r = 3$

When $r = 3$, the ordered Turán numbers are equivalent to an edge-labeling problem on the ordered complete graph $K_n^{(2)}$. A $k$-edge-labeling of $K_n^{(2)}$ assigns to each pair $uv$ a label in a linearly ordered set $S$ with $|S| = k$. Let $\phi$ be a $k$-edge-labeling of $K_n^{(2)}$. A triple of vertices $\{u, v, w\}$ with $u < v < w$ is good if $\phi(uv) < \phi(vw)$. For convenience, we write $uvw$ for the triple $\{u, v, w\}$. A triple is bad if it is not good. Let $f(n, k)$ be the maximum, over all $k$-edge-labelings $\phi$ of $K_n^{(2)}$, of the number of good triples. A $k$-edge-labeling $\phi$ of $K_n^{(2)}$ is optimal if it has $f(n, k)$ good triples.

Proposition 7. For $s \geq 3$, we have $\hat{e}(n, P_s^{(3)}) = f(n, s - 2)$.

Proof. First, we show $\hat{e}(n, P_s^{(3)}) \geq f(n, s - 2)$. Let $\phi$ be an $(s - 2)$-edge-labeling of $K_n^{(2)}$ with $f(n, s - 2)$ good triples. Let $G$ be the ordered 3-uniform hypergraph with vertex set $V(K_n^{(2)})$ such that for $u < v < w$, we have $uvw \in E(G)$ if and only if $\phi(uv) < \phi(vw)$. We claim that $G$ does not contain $P_s^{(3)}$. Indeed, if $v_1 \cdots v_s$ is a copy of $P_s^{(3)}$ in $G$, then $\phi(v_i v_{i+1}) < \phi(v_{i+1} v_{i+2})$ for $1 < i < s$ by definition of $G$. It follows that $\phi$ uses $s - 1$ distinct labels on the consecutive pairs of $v_1 \cdots v_s$, contradicting that $\phi$ is a $(s - 2)$-edge-labeling. It follows that $\hat{e}(n, P_s^{(3)}) \geq |E(G)| = f(n, s - 2)$.

Next, we show $\hat{e}(n, P_s^{(3)}) \leq f(n, s - 2)$. Let $G$ be a 3-uniform ordered hypergraph not containing $P_s^{(3)}$, and let $\phi$ be the edge-labeling on $V(G)$ by setting $\phi(uv)$, where $u < v$, equal to the length of a longest tight ordered path in $G$ that ends in $uv$. Clearly, if $u < v < w$ and $uvw \in E(G)$, then we have $\phi(uv) < \phi(vw)$ since the edge $uvw$ can be used to extend a longest ordered path ending in $uv$ to a longest path of larger length ending in $vw$. Therefore $\phi$ has at least $|E(G)|$ good triples. Note that $\phi$ assigns each pair $uv$ a value in the set $\{0, \ldots, s - 3\}$, since every ordered tight path in $G$ has at most $s - 1$ vertices and at most $s - 3$ edges. Since $\phi$ assigns each edge a value in $\{0, \ldots, s - 3\}$, it follows that $\phi$ is an $(s - 2)$-edge-labeling, and therefore $f(n, s - 2) \geq |E(G)| = \hat{e}(n, P_s^{(3)})$.

Next, we give lower bound constructions for $f(n, k)$ which we conjecture to be asymptotically optimal. The construction is easiest to describe when $k$ is odd. A labeling $\phi$ is monotone if, for all $u < v < w$, we have $\phi(uv) \leq \phi(vw)$.

Proposition 8. If $k$ is odd, then $f(n, k) \geq \left(1 - \frac{4}{(k+1)^2} + o(1)\right)\left(\binom{n}{3}\right)$.

Proof. Let $t = (k + 1)/2$. Let $(X_1, \ldots, X_t)$ be an interval partition of $K_n^{(2)}$ into $t$ parts whose sizes differ by at most 1. For $u < v$ with $u \in X_i$ and $v \in X_j$, we set $\phi(uv) = i + j$. Clearly, the range of $\phi$ is contained in $\{2, \ldots, 2t\}$, and so $\phi$ is a $(2t - 1)$-edge-labeling. Note that $2t - 1 = k$.

Since $\phi$ is a monotone labeling, the only triples $uvw$ with $u < v < w$ that are not good are those with $\phi(uv) = \phi(vw)$. Each such triple is contained in a part $X_i$ for some $i$. It follows that the number of triples that are not good is asymptotically equal to $t\left(\binom{n}{3}/t\right)$, which is asymptotically equal to $1/t\left(\binom{n}{3}\right)$. The proposition follows.

Corollary 9. If $s \geq 3$ and $s$ is odd, then $\hat{e}(n, P_s^{(3)}) \geq \left(1 - \frac{4}{(s-1)^2} + o(1)\right)\left(\binom{n}{3}\right)$.

Applying Proposition 7 to the construction in Proposition 8 gives a graph $G$ whose complement is the union of $t$ complete graphs on $t$ disjoint intervals of nearly equal size. The construction for even $k$ is more subtle.
We first give our construction in terms of a general interval partition \((X_1, \ldots, X_k)\) of \([n]\) into \(k\) parts. Later, we specify the sizes of the parts. For a pair \(uv\) with \(u < v, u \in X_i, v \in X_j\), we define \(\phi(uv)\) as follows. If \(i = j\), then \(\phi(uv) = i\). If \(i = j + 1\) or \(i = j - 1\), then we set \(\phi(uv)\) so that \(i < \phi(uv) < j\). Otherwise, \(j = i + 1\). The fractional index of \(u\) in \(X_i\) is \((u + 1 - \min X_j)/|X_i|\). Note that the fractional index of \(u\) is a real number in \((0, 1]\).

Let \(\lambda_u\) and \(\lambda_v\) be the fractional indices of \(u\) in \(X_i\) and \(v\) in \(X_{i+1}\), respectively. We set \(\phi(uv) = i\) if \(\lambda_u + \lambda_v \leq 1\) and \(\phi(uv) = i + 1\) if \(\lambda_u + \lambda_v > 1\).

**Lemma 10.** Let \(a, b, c\) be constants. If \(|X_{i-1}| = (a + o(1))n, |X_i| = (b + o(1))n,\) and \(|X_{i+1}| = (c + o(1))n\), then the number of bad triples \(uvw\) with \(v \in X_i\), \(u \in X_{i-1}\), \(w \in X_{i+1}\) is \((a + b)(b + c) + o(1)|X_i|\).

**Proof.** Let \(v \in X_i\) and let \(\lambda_v\) be the fractional index of \(v\) in \(X_i\). If \(uvw\) is a bad triple with \(u < v < w\), then \(u \in X_{i-1} \cup X_i, w \in X_i \cup X_{i+1}\), and \(\phi(uv) = \phi(uvw) = i\). For \(u \in X_i\), we require only that \(u\) precede \(v\), and there are \(\lambda_u|X_i| - 1\) such vertices in \(X_i\). For \(u \in X_{i-1}\), the condition \(\phi(uv) = i\) is equivalent to \(\lambda_u + \lambda_v > 1\), where \(\lambda_u\) is the fractional index of \(u\) in \(X_{i-1}\). The number of \(u \in X_{i-1}\) with \(\lambda_u + \lambda_v > 1\) is \([|X_{i-1}| - (1 - \lambda_u)|X_{i-1}|]\) or \([\lambda_u|X_{i-1}|]\). It follows that the number of choices for \(u\) in a bad triple \(uvw\) with \(u < v < w\) equals \(\lambda_u(|X_{i-1}| + |X_i|) + O(1)\).

Similarly, the number of \(w \in X_i\) that follow \(v\) is \((1 - \lambda_v)|X_i|\) and the number of \(w \in X_{i+1}\) with \(\lambda_u + \lambda_w \leq 1\) is \((1 - \lambda_v)|X_{i+1}|\). It follows that the number of choices for \(w\) in a bad triple \(uvw\) with \(u < v < w\) equals \((1 - \lambda_v)(|X_i| + |X_{i+1}|) + O(1)\).

Multiplying the number of choices for \(u\) and the number of choices for \(w\) gives a total of \(\lambda_u(1 - \lambda_v)(|X_{i-1}| + |X_i|) + O(n)\) bad triples \(uvw\) with \(u < v < w\). Suppose that \(X_i = \{v_1, \ldots, v_t\}\). Summing over all \(v \in X_i\), the total number of bad triples is \(O(n^2) + (|X_{i-1}| + |X_i|)(|X_i| + |X_{i+1}|) \sum_{j=1}^t (t - j)\), or \(O(n^2) + (|X_{i-1}| + |X_i|)(|X_i| + |X_{i+1}|) \sum_{j=1}^t (t + 1)\). Recalling that \(t = |X_i| = (b + o(1))n\), the number of bad triples \(uvw\) with \(u < v < w\) simplifies to \(O(n^2) + (a + b)(b + c)h(b + o(1))n^3\) and the lemma follows.

**Lemma 11.** If \(k\) is even, then \(f(n, k) \geq \left(1 - \frac{4}{k(k+2)} - o(1)\right)\frac{n^3}{3}\).

**Proof.** Suppose \(k\) is even, and let \(t = k/2\). Let \((Y_1, \ldots, Y_t)\) and \((Z_1, \ldots, Z_{t+1})\) be interval partitions of \(V(K_n^{(2)})\) into parts of nearly equal size, and let \((X_1, \ldots, X_k)\) be their common refinement. (Note that \((Y_1, \ldots, Y_t)\) is the partition used in our construction with \(k - 1\) labels, and \((Z_1, \ldots, Z_{t+1})\) is the partition used in our construction with \(k + 1\) labels.) For \(1 \leq j \leq k\), we set \(a_j\) equal to the limit of \(|X_j|/n\) as \(n \to \infty\). It is convenient to introduce \(X_0 = X_{t+1} = \emptyset\) and \(a_0 = a_{t+1} = 0\). When divided by \(n\) to normalize, the boundaries of \((Y_1, \ldots, Y_t)\) are \(\frac{k}{t+1}, \frac{1}{t+1}, \frac{2}{t+1}, \ldots, \frac{t}{t+1}\), and the boundaries of \((Z_1, \ldots, Z_{t+1})\) are \(\frac{0}{t+1}, \frac{1}{t+1}, \frac{1}{t+1}, \ldots, \frac{t+1}{t+1}\). In the common refinement, these boundaries interleave and are thus \(\frac{0}{t+1}, \frac{1}{t+1}, \frac{1}{t+1}, \frac{2}{t+1}, \frac{1}{t+1}, \ldots, \frac{t+1}{t+1}\). It follows that for \(0 \leq j \leq t\), we have \(a_{2j} = \frac{t - j}{t(t + 1)}\) and for \(0 \leq j \leq t\), we have \(a_{2j+1} = \frac{j}{t(t + 1)}\).

Let \(\phi\) be the labeling described above. Since \(k\) is constant, by Lemma 10, the number of bad triples is asymptotically \(\sum_{i=1}^k (a_i - 1 + a_i) a_i (a_i + a_{i+1}) \frac{n^3}{3}\). Let \(A\) be this sum taken over even indices \(i\), and let \(B\) be the sum taken over odd indices \(i\). We compute

\[
A = \sum_{j=1}^t (a_{2j-1} + a_{2j}) a_{2j} (a_{2j} + a_{2j+1}) \cdot \frac{n^3}{3} \\
= \left(\frac{n}{3}\right) \sum_{j=1}^t \left(\frac{t - (j - 1) + j}{t(t + 1)}\right) \cdot j + (t - j) \cdot \frac{j}{t(t + 1)} \\
= \left(\frac{n}{3}\right) \frac{1}{t(t + 1)} \sum_{j=1}^t j \\
= \frac{1}{2t(t + 1)} \frac{n^3}{3}
\]

The computation for \(B\) is symmetric and also gives \(B = \frac{1}{2t(t + 1)} \frac{n^3}{3}\). Since the total number of bad triples is asymptotic to \(A + B\), we have \(f(n, k) \geq \left(1 - \frac{1}{t(t + 1)} - o(1)\right)\frac{n^3}{3}\).
Corollary 12. If $s \geq 2$ and $s$ is even, then $\hat{e}x(n, P_s^{(3)}) \geq \left(1 - \frac{1}{(s-2)^2} + o(1)\right) \binom{n}{3}$.

We conjecture that these constructions are asymptotically optimal.

Conjecture 13. If $a = \lfloor (k+1)^2/4 \rfloor$, then $f(n, k) = \left(1 - \frac{1}{a} + o(1)\right) \binom{n}{3}$. Equivalently, if $b = \lfloor (s-1)^2/4 \rfloor$, then $\hat{e}x(n, P_s^{(3)}) = \left(1 - \frac{1}{b} + o(1)\right) \binom{n}{3}$.

Our goal in the remainder of this section is to show that if $k$ is odd and $\phi$ is monotone, then $\phi$ has at least $\left(\frac{1}{(k+1)^2} - o(1)\right) \binom{n}{3}$ bad triples. This shows that the construction in Proposition 8 is asymptotically optimal within the class of monotone labelings. For an edge-labeling $\phi$, let $\text{cost}(\phi)$ be the number of bad triples.

We believe that there exists an optimal monotone labeling of $K_n^{(2)}$. It is at least true that when $k \geq 2$, an optimal $[k]$-edge-labeling $\phi$ of $K_n^{(2)}$ does not contain a triple $uvw$ with $u < v < w$, $\phi(uv) = k$ and $\phi(vw) = 1$. Indeed, if there is such a triple, then we may fix $v$ and assume that $u$ is the minimum vertex such that $\phi(uv) = k$ and $w$ is the maximum vertex such that $\phi(vw) = 1$. Modify $\phi$ to obtain $\phi'$ by setting $\phi'(uv) = 1$ and $\phi'(vw) = k$. We have $\text{cost}(\phi') = \text{cost}(\phi) + a - b$, where $a$ is the number of triples that are good in $\phi$ but bad in $\phi'$, and $b$ is the number of triples that are bad in $\phi$ but good in $\phi'$. We show that $b > a$, contradicting that $\phi$ is optimal. By maximality of $w$, for each $u'$ with $w' > w$, the triple $uw'v'$ contributes to $b$. Similarly by minimality of $u$, every triple $u'vw$ with $u' < u$ also contributes to $b$. Also, the triple $uvw$ itself contributes to $b$. Therefore $b \geq (n-w) - (u-1) + 1$. Every triple contributing to $b$ contains $uv$ (and therefore has the form $u'vw$ for some $u' < u$), or $wv$ (and therefore has the form $vwv'$ for some $v' > v$). It follows that $a \leq (u-1) + (n-w) - (u-1) + 1 \leq b$.

For a monotone $[k]$-edge-labeling $\phi$ of $K_n^{(2)}$ define $\Phi_L$ and $\Phi_R$ as follows. We set $\Phi_L(1) = 0$ and $\Phi_L(v) = \max\{\phi(uv) : u < v\}$ for $v > 1$. We set $\Phi_R(n) = k+1$ and $\Phi_R(v) = \min\{\phi(vw) : w > v\}$ for $v < n$. Note that by monotonicity of $\phi$, if $u < v$, then $\Phi_L(u) \leq \Phi_R(u) \leq \phi(uv) \leq \Phi_L(v) \leq \Phi_R(v)$.

Using $\Phi_L$ and $\Phi_R$, we construct two interval partitions of $V(K_n^{(2)})$. For $0 \leq i \leq k+1$, let $X_i = \{v \in V(K_n^{(2)}) : \Phi_L(v) = i\}$ and let $\tilde{X}_i = \{v \in V(K_n^{(2)}) : \Phi_R(v) = i\}$. Note that $X_0 = \{1\}$, $\tilde{X}_0 = \emptyset$, $X_{k+1} = \emptyset$, and $\tilde{X}_{k+1} = \{n\}$. Since $\Phi_L$ and $\Phi_R$ inherit the monotonicity of $\phi$, both $(X_0, \ldots, X_{k+1})$ and $(\tilde{X}_0, \ldots, \tilde{X}_{k+1})$ are interval partitions of $V(K_n^{(2)})$. Our next lemma shows that these partitions are very similar.

Lemma 14. The symmetric difference of $X_i$ and $\tilde{X}_i$ has size at most 2, and so $||X_i| - |\tilde{X}_i|| \leq 2$.

Proof. Let $u, v \in X_i$ with $u < v$. Since $i = \Phi_L(u) \leq \Phi_R(u) \leq \phi(uv) \leq \Phi_L(v) = i$ it follows that $\Phi_R(u) = i$, and so $u \in X_i$. Therefore $X_i - X_i \subseteq \{\max X_i\}$. Similarly, $X_i - X_i \subseteq \{\min X_i\}$. □

A key step in our proof is the following bound on the sizes of the parts.

Lemma 15. Let $\phi$ be an monotone $[k]$-edge-labeling of $K_n^{(2)}$ that minimizes the number of bad triples, and define $\Phi_L$ and $(X_0, \ldots, X_{k+1})$ as above. For $2 \leq i \leq k$, we have $|X_i| + |X_{i+1}| \leq |X_{i-1}| + |X_{i-1} + 2$.

Proof. First, suppose that $X_i$ is nonempty. Let $v = \min X_i$ and let $u$ be the least vertex such that $\phi(uv) = i$. (Note that $u$ exists since $\Phi_L(v) = i > 0$.) If it exists, let $w$ be the least vertex in $X_{i+1}$ with $\phi(vw) = i + 1$. We say that an edge is long if its endpoints are in distinct non-consecutive parts in $(X_0, \ldots, X_{k+1})$.

Obtain $\phi'$ from $\phi$ by reducing by 1 the labels on $vw$ (if $w$ exists), $uv$, all long edges $u'u$ such that $u' < u$ and $\phi(u'u) = i - 1$, and all long edges $v'v$ such that $v' < v$ and $\phi(v'v) = i$. Note that $\phi'$ is a monotone labeling. We have $\text{cost}(\phi') = \text{cost}(\phi) + a - b$, where $a$ is the number of triples $xyz$ with $x < y < z$ which are bad in $\phi'$ and good in $\phi$, and $b$ is the number of triples $xyz$ with $x < y < z$ which are good in $\phi'$ and bad in $\phi$. Note that if $xyz$ is a triple with $x < y < z$ and $\phi$ and $\phi'$ agree on both $xy$ and $yz$, then of course $xyz$ contributes to neither $a$ nor $b$. Also, if $\phi$ and $\phi'$ disagree on $xy$ and $yz$, then also $xyz$ does not contribute to $a$ or $b$ since $\phi'(xy) = \phi(xy) - 1$ and $\phi'(yz) = \phi(yz) - 1$. So each triple $xyz$ contributing to $a$ or $b$ has one pair where $\phi$ and $\phi'$ agree, and one pair where $\phi$ and $\phi'$ disagree. Suppose that $xyz$ contributes to $a$. It follows that $\phi'(xy) = \phi(xy)$ and $\phi'(yz) = \phi(yz) - 1 = \phi(xy)$. Note that $y$ is not a long edge $u'u$, since $\phi(u'u) \leq \Phi_L(u') \leq \Phi_L(u) - 2 \leq i - 3$ and $\phi(u'u) = i - 1$, implying that $uvu$ is still good in $\phi'$. Similarly, $yz$ is not a long edge $v'v$. So $yz \in \{uv, vw\}$.
If $xyz = xuv$, then $\phi(xu) = \phi'(xu) = \phi'(uv) = i - 1$ and hence $u \in X_{i-1}$. It follows that $x \in X_{i-2} \cup X_{i-1}$ since $xu$ is not a long edge. So $xyz = xuv$ implies that $x < u$ and $x \in X_{i-2} \cup X_{i-1}$.

If $xyz = xuv$, then $\phi(xv) = \phi'(xv) = \phi'(uv) = i$. Since $u$ is the minimum vertex with $\phi(uv) = i$, we have $u \leq x$. Moreover, $xyz \neq uvw$ since $\phi$ and $\phi'$ both disagree on $uv$ and $vw$. Hence $u < x < v$. Also, $x \in X_{i-1}$ since $\phi(xv) = \phi'(xv)$, and so $xv$ is not a long edge with label $i$. Combining both cases, the contributions $xyz$ to $a$ arise from distinct $x \in X_{i-2} \cup X_{i-1}$, and it follows that $a \leq |X_{i-2}| + |X_{i-1}|$.

It remains to show that $b \geq |X_{i-1}| + |X_{i+1}| - 2$. Let $z \in X_i \cup X_{i+1}$ such that $z > v$ and $z \neq w$ (if $w$ exists). Note that $\phi(vz) = \phi'(vz)$, since $vw$ is the only edge incident to the right of $v$ where $\phi$ and $\phi'$ disagree. If $\phi(vz) = i$, then $uvw$ contributes to $b$ since $\phi(uv) = i$ but $\phi'(uv) = i - 1$. If $\phi(vz) > i$, then $i < \phi(vz) \leq \Phi_L(z) \leq i + 1$ and so $z \in X_{i+1}$. Hence $w$ exists and by minimality of $w$ we have that $w < z$. Since $\phi(vw) = i + 1$ and also $i + 1 = \Phi_L(w) \leq \phi(wz) \leq \Phi_L(z) \leq i + 1$, we have also $\phi(wz) = i + 1$. But $\phi'(vw) = i$ and $\phi'(wz) = \phi'(wz) = i + 1$, and it follows that $uvwz$ contributes to $b$. Therefore $b \geq |X_i| + |X_{i+1}| - 2$.

Since $\phi$ minimizes the number of bad triples among monotone labelings, it follows that $\text{cost}(\phi') \geq \text{cost}(\phi)$, giving $a \geq b$.

Finally, we consider the case that $X_i = \emptyset$. If $X_{i+1} = \emptyset$ also, then the inequality holds trivially. Let $v = \min\{X_{i+1}\}$. Since $\Phi_L(v) = i + 1$, there is a vertex $u$ such that $u < v$ and $\phi(uv) = i + 1$. Since $X_i = \emptyset$, it follows that $\Phi_L(u) \leq i - 1$, implying that $uv$ is a long edge. Obtain $\phi'$ from $\phi$ by reducing $\phi(uv)$ by 1, and leaving all other labels the same. As before, we have $\text{cost}(\phi') = \text{cost}(\phi) + a - b$, where $a$ is the number of triples $xyz$ that are good in $\phi$ and bad in $\phi'$, and $b$ is the number of triples $xyz$ that are bad in $\phi$ and good in $\phi'$. A contribution $xyz$ to $a$ requires that $yz = uv$ and $\phi'(xu) = \phi'(uv) = i$, but $\phi'(xu) = \phi(xu) \leq \Phi_L(xu) \leq i - 1$, and so $a = 0$. Also, if $z > w$ and $z \in X_{i+1}$, then $uvz$ contributes to $b$ since $\phi'(uv) = \phi'(uv) = i - 1 + 1$ and $\phi'(uv) = \phi'(uv) = i - 1$. It follows that $b \geq |X_{i+1}| - 1$ and by optimality of $\phi$, we obtain $0 = a \geq b \geq |X_{i+1}| - 1$, and the inequality follows.

**Corollary 16.** Let $\phi$ be a monotone $[k]$-edge-labeling of $\tilde{K}^{(2)}_n$ that minimizes the number of bad triples, and define $\Phi_R$ and $(X_0, \ldots, X_{k+1})$ as above. For $1 \leq j \leq k - 1$, we have $|X_{j-1}| + |X_j| \leq |X_{j+1}| + |X_{j+2}| + 2$.

**Proof.** Obtain $\phi'$ from $\phi$ by reversing the order of vertices in $\tilde{K}^{(2)}_n$ and inverting the labels, so that $\phi'(uv) = (k + 1) - \phi(v^*u^*)$, where $v^* = (n + 1) - v$ and $u^* = (n + 1) - u$. Note that $\phi'$ is a monotone $[k]$-edge-labeling with $\text{cost}(\phi') = \text{cost}(\phi)$. Moreover, defining $\Phi_L'$ with respect to $\phi'$ and the corresponding partition $(X_0', \ldots, X_{k+1})$, we have that $\Phi_L'(u) = (k + 1) - \Phi_R(u^*)$ where $u^* = (n + 1) - u$ and $|X_i'| = |X_{(k+1) - i}|$. Applying Lemma 15 to $\phi'$ with $i = (k + 1) - j$ gives the result.

**Theorem 17.** Let $\phi$ be a monotone $[k]$-edge-labeling of $\tilde{K}^{(2)}_n$ that minimizes the number of bad triples. If $k$ is odd, then $\text{cost}(\phi) \geq (1 - o(1)) \frac{4}{(k+1)^2} \binom{n}{2}$.

**Proof.** Define $\Phi_L, \Phi_R, (X_0, \ldots, X_{k+1})$, and $(\hat{X}_0, \ldots, \hat{X}_{k+1})$ as above, and let $m = (k - 1)/2$. For $0 \leq \ell \leq m$, let $a_{\ell} = |X_{2\ell}| + |X_{2\ell + 1}|$ and let $a_{\ell} = |\hat{X}_{2\ell}| + |\hat{X}_{2\ell + 1}|$. By Lemma 15 with $i = 2\ell$, we have $a_{\ell-1} \geq a_\ell - 2$ for $1 \leq \ell \leq m$. It follows that $a_\ell \geq a_1 - 2 \geq \cdots \geq a_m - 2m = a_m - (k - 1)$. With two applications of Lemma 14, we have $a_\ell \leq \hat{a}_0 + 4$. By Corollary 16 with $j = \ell - 1$, we have $\hat{a}_{\ell-1} \leq \hat{a}_0 + 2$ for $1 \leq \ell \leq m$. It follows that $\hat{a}_0 \leq \hat{a}_0 + 2 \leq \cdots \leq \hat{a}_m + 2m = \hat{a}_m + (k - 1)$. Chaining the inequalities gives $-(k - 1) + a_m \leq \cdots \leq a_0 \leq \hat{a}_0 + 4 \leq \cdots \leq \hat{a}_m + k + 3 \leq a_m + k + 7$. It follows that $-2\ell + a_\ell$ is in the interval $[a_m - (k - 1), a_m + (k + 7)]$, and so $a_\ell$ is in the interval $[a_m - (k - 1), a_m + (k + 7) + 2\ell]$. Since $2\ell \leq k - 1$, we have $|a_m - a_\ell| \leq 2k + 6$ and so each pair in $(a_0, \ldots, a_m)$ differs by at most $4k + 12$. Since $\sum_{\ell=0}^m a_\ell = n$, it follows that $a_\ell = n/(m+1) + O(k)$.

Similarly, for $0 \leq \ell \leq m$, let $b_\ell = |X_{2\ell+1}| + |X_{2\ell+2}|$ and let $b_\ell = |\hat{X}_{2\ell+1}| + |\hat{X}_{2\ell+2}|$. By Lemma 15 with $i = 2\ell + 1$, we have $b_{\ell-1} \geq b_\ell - 2$ for $1 \leq \ell \leq m$. It follows that $b_0 \geq b_1 - 2 \geq \cdots \geq b_m - (k - 1)$. Similarly, $b_0 \leq \hat{b}_0 + 4$. Also, Corollary 16 with $j = 2\ell$ gives $\hat{b}_{\ell-1} \leq \hat{b}_\ell + 2$ for $1 \leq \ell \leq m$. Therefore $b_0 \leq b_1 + 2 \leq \cdots \leq b_m + (k - 1)$. Combining the inequalities gives $-(k - 1) + b_m \leq \cdots \leq b_0 \leq b_0 + 4 \leq b_0 + (k + 3) \leq b_m + (k + 7)$. As above, $b_\ell$ and $b_{\ell'}$ differ by at most $4k + 12$. Again, $\sum_{\ell=0}^m b_\ell = n$, and so $b_\ell = n/(m+1) + O(k)$.

Let $0 \leq \ell \leq m$. We claim that $X_{2\ell}$ and $X_{2\ell+2}$ differ in size by at most $O(k)$. Indeed, both $a_\ell$ and $b_\ell$ equal $n/(m+1) + O(k)$, and so $|a_\ell - b_\ell| \leq O(k)$. Since $a_\ell = |X_{2\ell}| + |X_{2\ell+1}|$ and $b_\ell = |X_{2\ell+1}| + |X_{2\ell+2}|$, we have
that \(a_\ell - b_\ell = |X_{2\ell}| - |X_{2\ell+2}|\) and the claim follows. Therefore each pair of parts in \(\{X_0, X_2, X_4, \ldots, X_{k+1}\}\) differs in size by at most \(O(k^2)\). Since \(|X_{k+1}| = 0\), it follows that each part with even index has size at most \(O(k^2)\). Since \(a_\ell = |X_{2\ell}| + |X_{2\ell+1}| = n/(m + 1) + O(k)\), it follows that each part with odd index has size \(n/(m + 1) - O(k^2)\). Since each triple \(uvw\) with \(u, v, w \in X_i\) satisfies \(\phi(uv) = \phi(vw) = i\), the number of bad triples in \(\phi\) is at least \(\sum_{\ell=0}^{m} \left(\frac{|X_{2\ell+1}|}{3}\right)\), and \(\sum_{\ell=0}^{m} \left(\frac{|X_{2\ell+1}|}{3}\right) \geq (m + 1)\left(\frac{n/(m+1) - O(k^2)}{3}\right) = (1 - o(1))\left(\frac{1}{m+1}\right)^2 \left(\frac{n}{3}\right).

Let \(k\) be even, let \(\phi\) be an optimal monotone \([k]\)-edge-labeling of \(K_n^{(2)}\), and define the parts \((X_0, \ldots, X_{k+1})\) as above. Similar arguments as in Theorem 17 can be used to obtain the sizes of the parts asymptotically, and these match the sizes of the corresponding parts in our construction in Lemma 11. However, bounding the number of bad triples in \(\phi\) for even \(k\) is more complicated since consecutive parts \(X_i\) and \(X_{i+1}\) are both linear in \(n\), making the triples with two vertices in one of \(\{X_i, X_{i+1}\}\) and one in the other significant.

References


