Ordered Turán Numbers

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- Note: $\vec{K}_{1,2}^{(2)}, \vec{K}_{2,1}^{(2)} \subseteq \vec{K}_{3,3}^{(2)}$ but $\vec{P}_3^{(2)} \not\subseteq \vec{K}_{3,3}^{(2)}$.

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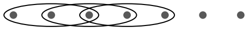

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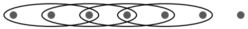

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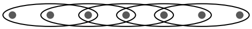

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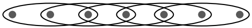

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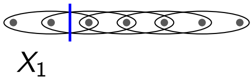

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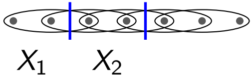

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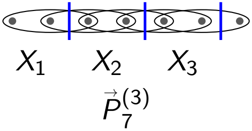

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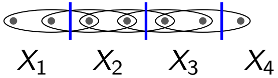

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- Thm (Győri–Korándi–Methuku–Tomon–Tompkins–Vizer 2018): $\mathrm{ex}(n, \mathcal{H}_k) = \Theta(n^{1+1/k})$, where $\mathcal{H}_k$ is the family of ordered cycles H on at most $2k$ vertices such that $\chi_i(H) = 2$ and $E(H)$ contains two particular edges.

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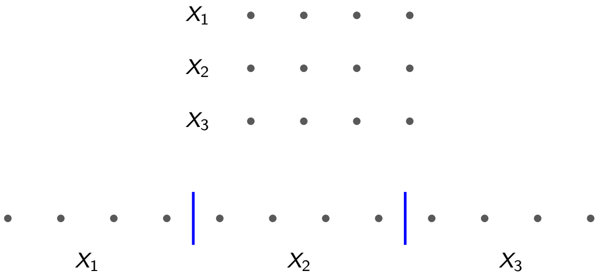

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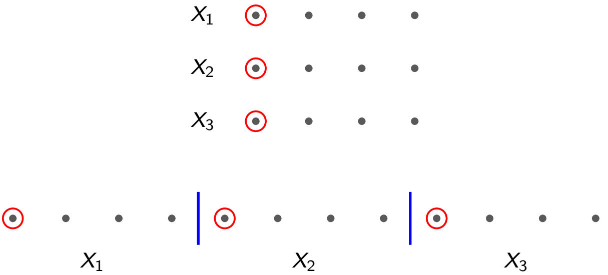

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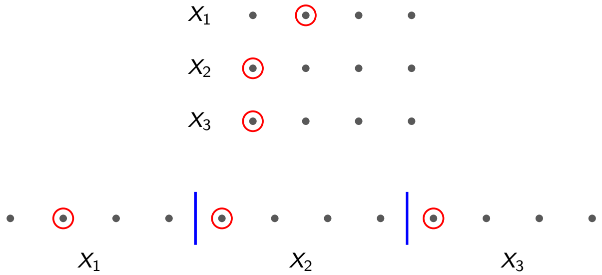

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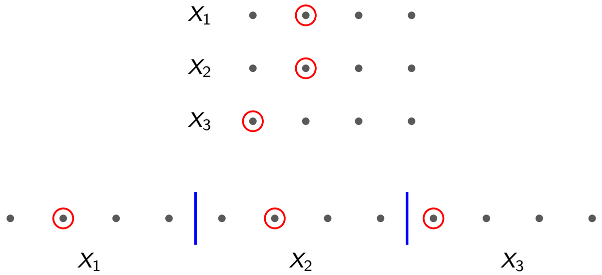

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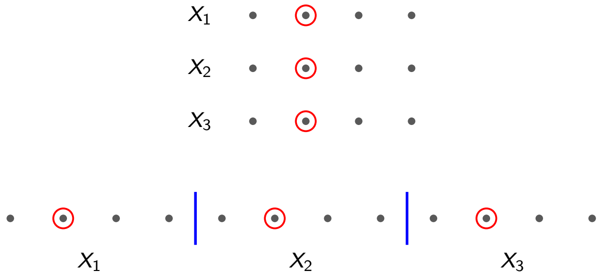

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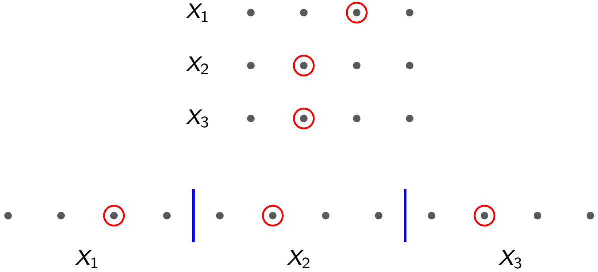

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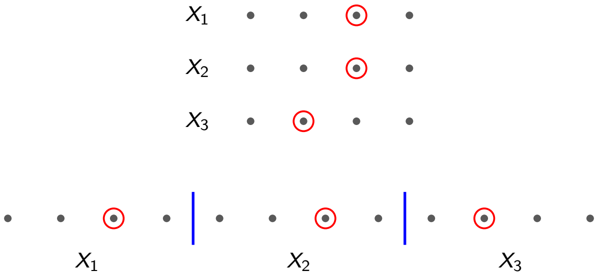

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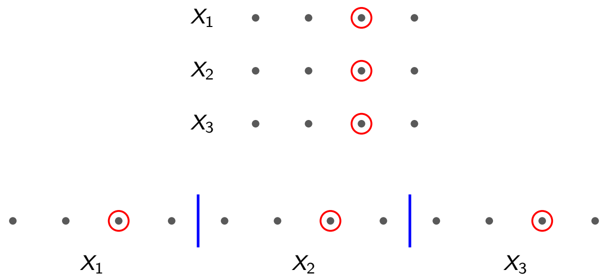

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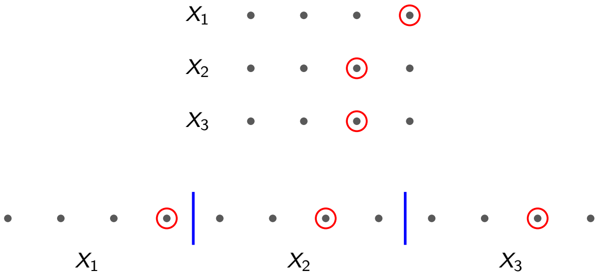

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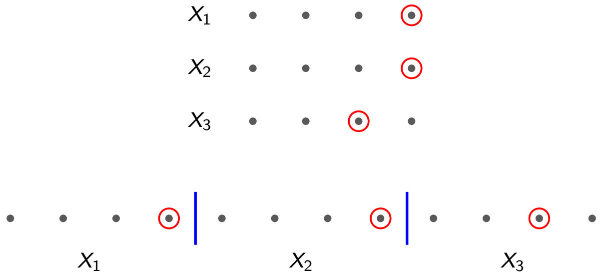

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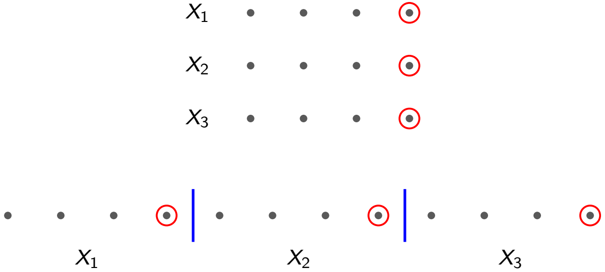

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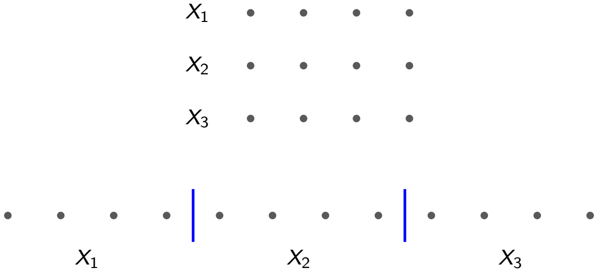

- Thm (Füredi–Jiang–Kostochka–Mubayi–Verstraëte 2021):

$$\vec{\mathrm{ex}}(n, \vec{Q}_s^{(r)}) = \begin{cases} \binom{n}{r} - \binom{n-(s-r)}{r} & \text{if } r \le s \le 2r \\ \Theta(n^{r-1}\log n) & \text{if } s > 2r \end{cases}$$

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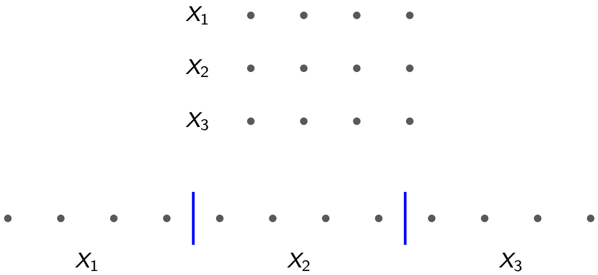

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Transversals and Packings

- Complementary transversal numbers for r-uniform H:

$$\vec{\tau}(n, H) = \min\{|E(G)| \colon G \subseteq \vec{K}_n^{(r)} \text{ and every copy of } H \text{ in } \vec{K}_n^{(r)} \text{ has an edge in } G\}$$

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- Always $\vec{\nu}(n,H) \leq \vec{\tau}(n,H)$.

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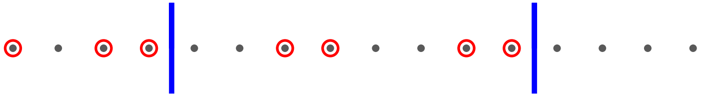

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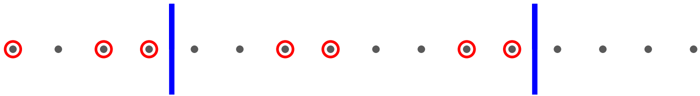

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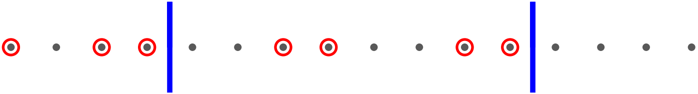

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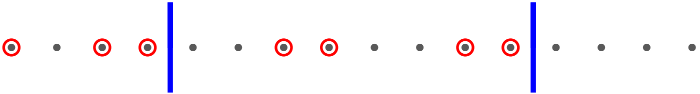

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- Define m-right-biased in the natural way. A set is m-biased if it is m-left-biased or m-right-biased.

Loose path transversals

$(r, s, m) = (4, 6, 3)$

Theorem

Let $r \leq s \leq 2r - 1$, let $m = s - r + 1$, and let n be even. We have $\vec{\tau}(n, \vec{LP}_s^{(r)}) \leq 2h(n, r, m) + h(n, r - 1, m)$.

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- Let $G \subseteq \vec{K}_n^{(r)}$ such that $E(G) = E_1 \cup E_2$.

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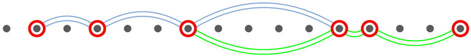

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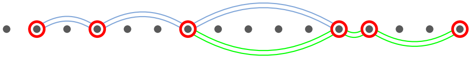

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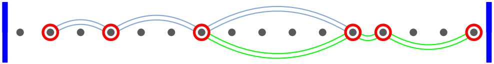

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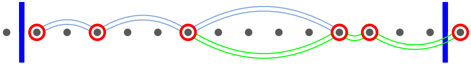

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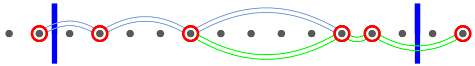

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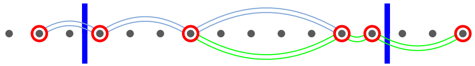

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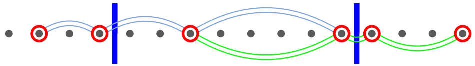

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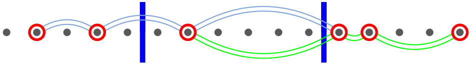

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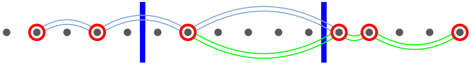

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- Such a partition exists as otherwise $s \leq 2(m - 1) = 2(s - r)$ and so $s \geq 2r$.

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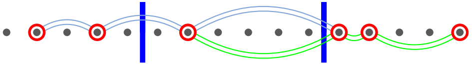

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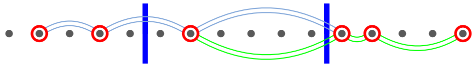

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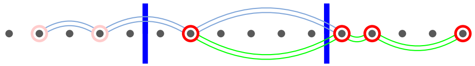

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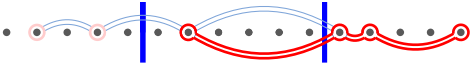

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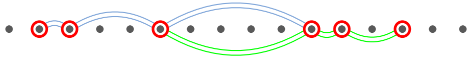

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Loose path transversals

$(r, s, m) = (4, 6, 3)$

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Let $r \leq s \leq 2r - 1$, let $m = s - r + 1$, and let n be even. We have $\vec{\tau}(n, \vec{LP}_s^{(r)}) \leq 2h(n, r, m) + h(n, r - 1, m)$.

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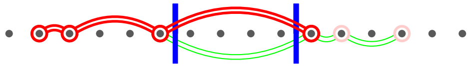

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Tight path packings

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Let $r \leq s \leq 2r-1$, let $m = s-r+1$, and let n be even. We have $\vec{\nu}(n, \vec{P}_s^{(r)}) \geq 2h(n,r,m) + h(n,r-1,m)$.

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- Show that $\{Q_e \colon e \in E(G)\}$ is edge-disjoint.

Fractional Variants

Frac. H packing $\vec{\nu}^*(n,H)$:

max: $$\sum_{H \subseteq \vec{K}_n^{(r)}} y_H$$

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- If $r \mid s$ and s is large, then $\displaystyle\lim_{n\to\infty} \frac{\vec{\tau}(n,\vec{P}_s^{(r)})}{\vec{\tau}^*(n,\vec{P}_s^{(r)})} > 1$.

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- We need a new approach for large s, probably even $s = 2r$.

The case $r = 3$

- Thm: $\vec{\tau}(n, \vec{P}_s^{(3)}) \leq \left(\frac{1}{\left\lfloor \frac{(s-1)^2}{4} \right\rfloor} + o(1) \right) \binom{n}{3}$

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Increasing Triple Optimization

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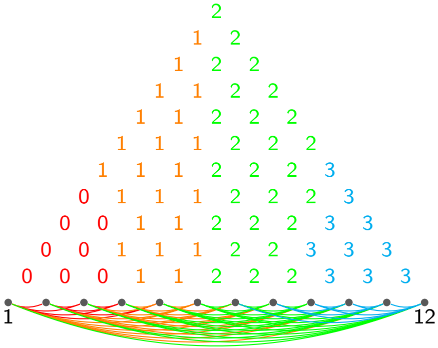
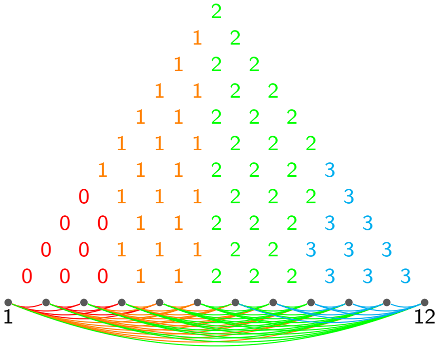
- First open case is $\vec{P}_6^{(3)}$, with 4 edge labels.
- Conjecture: $f_4(n) = \vec{\mathrm{ex}}(n, \vec{P}_6^{(3)}) = \left(\frac{5}{6} + o(1)\right)\binom{n}{3}$



Increasing Triple Optimization

- Let $f_k(n)$ be the max., over all $[k]$-edge-labelings of $\vec{K}_n^{(2)}$, of the number of triples $u < v < w$ with $\ell(uv) < \ell(vw)$.
- Prop: $\vec{\mathrm{ex}}(n, \vec{P}_s^{(3)}) = f_{s-2}(n)$.
- First open case is $\vec{P}_6^{(3)}$, with 4 edge labels.
- Conjecture: $f_4(n) = \vec{\mathrm{ex}}(n, \vec{P}_6^{(3)}) = \left(\frac{5}{6} + o(1)\right)\binom{n}{3}$



Thank You.