Abstract

We show that connected graphs admit sublinear longest path transversals. This improves an earlier result of Rautenbach and Sereni and is related to the fifty-year-old question of whether connected graphs admit longest path transversals of constant size. The same technique allows us to show that 2-connected graphs admit sublinear longest cycle transversals.

1 Introduction

A classical exercise in graph theory is to show that if \( P \) and \( Q \) are longest paths in a connected graph, then the vertex sets of \( P \) and \( Q \) have non-empty intersection (see [8], exercise 1.2.40). In 1966, Gallai [2] asked whether this result could be strengthened to assert that the family of all longest paths in a connected graph \( G \) has non-empty intersection. It turns out the answer is no, as shown by Walther [6] with a 25-vertex counterexample. A 12-vertex counterexample, due to Walther and Voss [7] and independently Zamfirescu [10], is obtained from the Petersen graph by replacing one vertex \( v \) with an independent set \( \{v_1, v_2, v_3\} \) such that each \( v_i \) becomes an endpoint of an edge incident to \( v \) (see Figure 1).

Since Gallai’s question has a negative answer, a single vertex is generally insufficient to meet every longest path in a connected graph \( G \). A longest path transversal in \( G \) is a set of vertices that intersects every longest path. Such a set is a transversal in the hypergraph on \( V(G) \) whose edges are the vertex sets of longest paths in \( G \). Let \( \text{lpt}(G) \) be the minimum size of a longest path transversal in \( G \). The graph \( G_0 \) in Figure 1 is a connected 12-vertex graph with \( \text{lpt}(G_0) = 2 \). Grünbaum [3] constructed a connected 324-vertex graph \( G \) with \( \text{lpt}(G) = 3 \). Soon afterward, Zamfirescu [10] found such a graph with 270 vertices. Walther [6] and Zamfirescu [9] asked if \( \text{lpt}(G) \) is bounded for connected graphs \( G \), and this remains
Figure 1: The graph $G_0$: a 12-vertex graph with $\lpt(G_0) = 2$.

open. In fact, it is not known whether there is a connected graph $G$ with $\lpt(G) \geq 4$. Let $G$ be a connected graph. Since a connected graph does not contain vertex-disjoint longest paths, every partition of $V(G)$ into two sets has a part that contains no longest path in $G$, forcing the other part to be a longest path transversal. Applying this to a partition of $V(G)$ into two parts of nearly equal size gives $\lpt(G) \leq \lceil n/2 \rceil$ when $G$ is an $n$-vertex connected graph. It is not too difficult to improve this argument to obtain $\lpt(G) \leq \lceil n/4 \rceil$. Rautenbach and Sereni [4] showed that $\lpt(G) \leq \lceil n/4 - n^{2/3} / 90 \rceil$ for every connected $n$-vertex graph $G$. We show that $\lpt(G) \leq 8n^{3/4}$ when $G$ is an $n$-vertex connected graph, implying that connected graphs have sublinear longest path transversals.

Let $\lct(G)$ be the minimum size of a set of vertices $S$ such that $S$ intersects every longest cycle in $G$. Analogously to the case of longest paths in 1-connected graphs, every pair of longest cycles in a 2-connected graph intersect. The Petersen graph $G$ is 2-connected and $\lct(G) = 2$. With no connectivity assumptions, Thomassen [5] showed that $\lct(G) \leq \lceil n/3 \rceil$ for each $n$-vertex graph $G$. The bound is sharp when $G$ is a disjoint union of triangles and nearly sharp in the 1-connected case when $G$ is obtained from a star with $(n-1)/3$ leaves by replacing each leaf with a triangle. On the other hand, Rautenbach and Sereni [4] proved that if $G$ is 2-connected, then $\lct(G) \leq \lceil n/3 - n^{2/3} / 36 \rceil$. We show that $\lct(G) \leq 20n^{3/4}$ when $G$ is 2-connected (Corollary 2).

The problems of finding small longest path transversals and small longest cycle transversals are special cases of a general problem that we aim to address. Given a multigraph $F$ and an edge $e \in E(F)$ with endpoints $u$ and $v$, the subdivision operation produces a new multigraph $F'$ in which $e$ is replaced by a path $uvw$ through a new vertex $w$ in $F'$. A subdivision of $F$ is a graph obtained from $F$ via a sequence of zero or more subdivision operations. For a multigraph $R$ and a graph $G$, an $R$-subdivision in $G$ is a subgraph of $G$ isomorphic to a subdivision of $R$. We ask for a small set of vertices in $G$ that intersects every $R$-subdivision in $G$ of maximum size. The cases of longest path transversals and longest cycle transversals arise as $R = P_2$ and $R = C_2$ (the multigraph 2-vertex cycle), respectively. We prove that for each connected multigraph $R$, if the family $\mathcal{F}$ of maximum $R$-subdivisions in $G$ is pairwise intersecting, then $\mathcal{F}$ admits a transversal of size at most $Cn^{3/4}$, where $C$ is a constant depending on $R$. 

2
2 Maximum subdivision transversals

Let $R$ be a multigraph. Recall that an $R$-subdivision in $G$ is a subgraph of $G$ isomorphic to a subdivision of $R$, and a maximum $R$-subdivision is an $R$-subdivision $F$ in $G$ that maximizes $|V(F)|$. An $R$-transversal of $G$ is a set of vertices intersecting each maximum $R$-subdivision. Let $\tau_R(G)$ be the minimum size of an $R$-transversal in $G$.

Given sets of vertices $X$ and $Y$ of $G$, an $(X,Y)$-separator is a set of vertices $S$ such that no path in $G - S$ has one endpoint in $X$ and the other endpoint in $Y$. We allow an $(X,Y)$-separator to contain vertices in $X$ and $Y$. An $(X,Y)$-connector is a collection of vertex-disjoint paths $\{P_1, \ldots, P_k\}$ such that each $P_i$ has one endpoint in $X$, the other endpoint in $Y$, and the interior vertices of $P_i$ are outside $X \cup Y$. A variant of Menger’s Theorem asserts that the minimum size of an $(X,Y)$-separator equals the maximum size of an $(X,Y)$-connector (see, e.g., Theorem 3.3.1 in [1]).

Our next result shows that when the maximum $R$-subdivisions in a graph $G$ pairwise intersect, $G$ has sublinear $R$-transversals. We make no attempt to optimize the multiplicative constant 8 or the dependence on $n$.

**Theorem 1.** Let $R$ be a connected $m$-edge multigraph with $m \geq 1$ and let $G$ be an $n$-vertex graph. If the maximum $R$-subdivisions in $G$ pairwise intersect, then $\tau_R(G) \leq 8m^{5/4}n^{3/4}$.

**Proof.** Let $m = |E(R)|$ and let $\varepsilon = 2(m/n)^{1/4}$. We may assume that $m \leq n$, since otherwise we may take $V(G)$ as our $R$-transversal. Let $\mathcal{F}$ be the family of maximum $R$-subdivisions in $G$. An $\varepsilon$-partial transversal is a triple $(H, X, Y)$ such that $H$ is a subgraph of $G$, $X = V(G) - V(H)$, $Y \subseteq X$ with $|Y| \leq \varepsilon |X|$, and each $F \in \mathcal{F}$ is a subgraph of $H$ or contains a vertex in $Y$. Given an $\varepsilon$-partial transversal $(H, X, Y)$, we either obtain an $\varepsilon$-partial transversal $(H', X', Y')$ with $|V(H')| < |V(H)|$ or we produce an $R$-transversal with at most $8m^{5/4}n^{3/4}$ vertices. Starting with $(H, X, Y) = (G, \emptyset, \emptyset)$ and iterating gives the result.

Let $(H, X, Y)$ be an $\varepsilon$-partial transversal, and let $\mathcal{F}_0$ be the set of $F \in \mathcal{F}$ such that $F$ is a subgraph of $H$. We may assume that $H$ contains vertex-disjoint paths $P_1$ and $P_2$ each of size $\lceil \varepsilon n \rceil$. Otherwise, every path in $H$ has size less than $2 \lceil \varepsilon n \rceil$, and so each $F \in \mathcal{F}_0$ has at most $2m \lceil \varepsilon n \rceil$ vertices. Since $\mathcal{F}_0$ is pairwise intersecting, we have that $V(F) \cup Y$ is an $R$-transversal for each $F \in \mathcal{F}_0$. It follows that $\tau_R(G) \leq |Y| + 2m \lceil \varepsilon n \rceil \leq \varepsilon n + 2m \lceil \varepsilon n \rceil \leq (2m + 1)\varepsilon n + 2m \leq (2m + 2)\varepsilon n \leq 4m \varepsilon n = 8m^{5/4}n^{3/4}$.

Suppose that $H$ has a $(V(P_1), V(P_2))$-separator $S$ of size at most $\varepsilon^2 n$. Since graphs in $\mathcal{F}_0$ are connected, each $F \in \mathcal{F}_0$ has a vertex in $S$ or is contained in some component of $H - S$. Also, since $\mathcal{F}_0$ is pairwise intersecting, at most one component $H'$ of $H - S$ contains graphs in $\mathcal{F}_0$. Since $S$ is a separator, $H'$ is disjoint from at least one of $\{P_1, P_2\}$. With $X' = V(G) - V(H')$ and $Y' = Y \cup S$, we have $|X'| - |X| \geq \varepsilon n$ and $|Y'| = |Y| + |S| \leq \varepsilon |X| + \varepsilon^2 n \leq \varepsilon |X| + \varepsilon (|X'| - |X|) \leq \varepsilon |X'|$. It follows that $(H', X', Y')$ is an $\varepsilon$-partial transversal. Also $|V(H')| < |V(H)|$ since $|X'| > |X|$.

Otherwise, by Menger’s Theorem, $H$ has a $(V(P_1), V(P_2))$-connector $\mathcal{P}$ with $|\mathcal{P}| \geq \varepsilon^2 n$. Let $\mathcal{P}'$ be the set of paths in $\mathcal{P}$ of size at most $2/e^2$. Note that $|\mathcal{P}'| \geq |\mathcal{P}|/2$, or else $\mathcal{P}$ has at least $(\varepsilon^2 n)/2$ paths of size more than $2/e^2$, contradicting that the paths in $\mathcal{P}$ are disjoint. So we have $|\mathcal{P}'| \geq |\mathcal{P}|/2 \geq (\varepsilon^2/2)n = 2m^{1/4}n^{1/2} \geq 2$. Combining $P_1$ with two paths in $\mathcal{P}'$ whose endpoints in $V(P_1)$ are as far apart as possible and a segment of $P_2$ gives a cycle $C_0$ such that $(\varepsilon^2/2)n \leq |V(C_0)| \leq 2 \lceil \varepsilon n \rceil + 4/\varepsilon^2 - 4 \leq 2\varepsilon n + 4/\varepsilon^2$, where the lower bound
counts vertices in $V(P_1) \cap V(C_0)$ and the upper bound counts at most $2 \lceil \varepsilon n \rceil$ vertices in $(V(P_1) \cup V(P_2)) \cap V(C_0)$, at most $4/\varepsilon^2$ vertices on the paths in $P'$ linking $P_1$ and $P_2$, and observing that the 4 endpoints of the linking paths are counted twice.

Let $C$ be a longest cycle in $H$ subject to $|V(C)| \leq 2\varepsilon n + 4/\varepsilon^2$, let $\ell = |V(C)|$, and note that $\ell \geq |V(C_0)| \geq (\varepsilon^2/2)n$. If $V(C)$ intersects each subgraph in $F_0$, then $Y \cup V(C)$ witnesses $\tau_R(G) \leq |V(C)| + |Y| \leq (2\varepsilon n + 4/\varepsilon^2) + \varepsilon n = 3\varepsilon n + (n/m)^{1/2} < 8m^{5/4}n^{3/4}$. Otherwise, choose $F \in F_0$ that is disjoint from $C$. We may assume $|V(F)| \geq \ell$, or else $Y \cup V(F)$ witnesses that $\tau_R(G) \leq |V(F)| + |Y| < (2\varepsilon n + 4/\varepsilon^2) + \varepsilon n < 8m^{5/4}n^{3/4}$.

If $H$ has a $(V(C), V(F))$-separator $T$ of size at most $\varepsilon \ell$, then we obtain an $\varepsilon$-partial transversal as follows. At most one component $H'$ of $H - T$ contains graphs in $F_0$. Let $X' = V(G) - V(H')$ and let $Y' = Y \cup T$. Since $H'$ is disjoint from one of $\{C, F\}$, it follows that $|X'| - |X| \geq \ell$. We compute $|Y'| = |Y| + |T| \leq \varepsilon |X| + \varepsilon \ell \leq \varepsilon |X| + \varepsilon (|X'| - |X|) \leq \varepsilon |X'|$. Hence $(H', X', Y')$ is an $\varepsilon$-partial transversal with $|V(H')| < |V(H)|$.

Otherwise, $H$ has a $(V(C), V(F))$-connector $Q$ with $|Q| \geq \varepsilon \ell$. We use $Q$ to obtain a contradiction. For $e \in E(R)$, let $Q_e$ be the path in $F$ corresponding to $e$, and let $Q_e$ be the set of paths in $Q$ which have an endpoint in $Q_e$. Since $|E(R)| = m$, it follows that $|Q_e| \geq |Q|/m \geq \varepsilon \ell/m$ for some edge $e \in E(R)$. Let $Q'$ be the set of paths in $Q_e$ of size at most $2mn \ell/e$, and note that $|Q'| \geq |Q_e|/2 \geq \ell/2m$, or else $Q_e$ has at least $\ell/2m$ paths of size more than $2mn \ell/e$, a contradiction. The endpoints of paths in $Q'$ divide $Q_e$ into $|Q'| - 1$ edge-disjoint subpaths. Choose $Q_1, Q_2 \in Q'$ to minimize the length of such a subpath $Q_0$ of $Q_e$, and note that $Q_0$ has length at most $\frac{n-1}{|Q|-1}$; see Figure 2. Since $m \leq n$, we have $2m \leq 2m^{3/4}n^{1/4} = \frac{\varepsilon^3}{4} n \leq \frac{\varepsilon^3}{2}$, and hence $\frac{n-1}{|Q|-1} \leq \frac{4m}{\varepsilon \ell - 2m} \leq \frac{4mn}{\varepsilon ^2}$. The endpoints of $Q_1$ and $Q_2$ on $C$ partition $C$ into two subpaths; let $W$ be the longer subpath. If $|E(W)| \geq |E(Q_0)|$, then we would obtain a larger $R$-subdivision by using $Q_1$, $W$, and $Q_2$ to bypass $Q_0$. Since $F$ is a maximum $R$-subdivision, we have $|E(W)| < |E(Q_0)|$. Therefore using $Q_1$, $Q_0$, and $Q_2$ to bypass $W$ gives a cycle $D$ with $|E(D)| > |E(C)|$. By the extremal choice of $C$, it follows that $|V(D)| > 2\varepsilon n + 4/\varepsilon^2$. On the other hand, $|V(D)| = \ldots$
Let maximum \( k \) and let maximum \( S \) for each \( R \) to obtain sublinear \( \tau \).

Therefore \( 2\varepsilon n + \frac{4}{\varepsilon^2} < |V(D)| \leq \frac{\ell}{2} + \frac{2mn}{\ell} \varepsilon^2 + \frac{4mn}{\ell^2} \varepsilon^2 + \frac{2mn}{\ell} \leq \frac{\ell}{2} + \frac{8mn}{\ell} \varepsilon^2 \), where the last inequality uses \( \ell \geq (\varepsilon^2/2)n \). Simplifying gives \( \varepsilon n < \frac{16m}{\varepsilon^2} - \frac{2}{\varepsilon^2} \leq \frac{16m}{\varepsilon^2} \), and this inequality is violated when \( \varepsilon \geq (16m/n)^{1/4} \).

Applying Theorem 1, we obtain the following corollary.

**Corollary 2.** Let \( G \) be an \( n \)-vertex graph. If \( G \) is connected, then \( \text{lpt}(G) \leq 8n^{3/4} \). If \( G \) is 2-connected, then \( \text{lct}(G) \leq 20n^{3/4} \).

**Proof.** When \( R = P_2 \), an \( R \)-transversal is a longest path transversal. It is well known that if \( G \) is connected, then the longest paths pairwise intersect. By Theorem 1, we have \( \text{lpt}(G) = \tau_R(G) \leq 8n^{3/4} \).

Similarly, when \( R = C_2 \), an \( R \)-transversal is a longest cycle transversal. If \( G \) is 2-connected, then the longest cycles pairwise intersect. By Theorem 1, we have \( \text{lct}(G) = \tau_R(G) \leq 8 \cdot 2^{5/4} \cdot n^{3/4} \leq 20n^{3/4} \).

We do not know whether the assumption in Theorem 1 that \( R \) is connected is necessary to obtain sublinear \( R \)-transversals. To obtain analogues of Corollary 2 for general \( R \), we show that the maximum \( R \)-subdivisions pairwise intersect when the connectivity of \( G \) is sufficiently large. Recall that a graph \( G \) is \( k \)-connected if \( |V(G)| > k \) and \( G - S \) is connected for each \( S \subseteq V(G) \) with \( |S| < k \). Moreover, the connectivity of \( G \), denoted \( \kappa(G) \), is the maximum \( k \) such that \( G \) is \( k \)-connected.

**Lemma 3.** Let \( R \) be a connected \( m \)-edge multigraph with \( m \geq 1 \). If \( \kappa(G) > m^2 \), then the maximum \( R \)-subdivisions in \( G \) are pairwise intersecting.

**Proof.** Suppose for a contradiction that \( G \) has disjoint maximum \( R \)-subdivisions \( F_1 \) and \( F_2 \), and let \( k = |V(F_1)| = |V(F_2)| \). By Menger’s Theorem, there is an \( (V(F_1), V(F_2)) \)-connector \( P \) with \( |P| = \min\{k, m^2 + 1\} \). If \( |P| = k \), then every vertex in \( F_1 \) is an endpoint of a path in \( P \), and we obtain an \( R \)-subdivision of size more than \( k \) by replacing an edge \( uv \in E(F_1) \) with a path in \( P \) having \( u \) as an endpoint, a path in \( P \) having \( v \) as an endpoint, and an appropriate path in the connected subgraph \( F_2 \).

So we may assume \( |P| = m^2 + 1 \). For each \( e \in E(R) \), let \( F_i(e) \) be the path in \( F_i \) corresponding to \( e \). Since \( R \) has no isolated vertices, we may associate each \( P \in P \) with an ordered pair of edges \( (e_1, e_2) \in (E(R))^2 \) such that \( P \) has its endpoint in \( F_1 \) in \( F_1(e_1) \) and its endpoint in \( F_2 \) in \( F_2(e_2) \). Since \( |P| > m^2 \), some pair \( (e_1, e_2) \) is associated with distinct paths \( P, Q \in P \). Let \( W_i \) be the subpath of \( F_i(e_i) \) whose endpoints are in \( V(P) \cup V(Q) \). If \( |E(W_1)| \geq |E(W_2)| \), then we modify \( F_2 \) to obtain a larger \( R \)-subdivision by using \( P, W_1 \), and \( Q \) to bypass \( W_2 \). Similarly, if \( |E(W_2)| \geq |E(W_1)| \), then we modify \( F_1 \) to obtain a larger \( R \)-subdivision by using \( P, W_2 \), and \( Q \) to bypass \( W_1 \).

**Corollary 4.** Let \( R \) be a connected \( m \)-edge multigraph. If \( G \) is an \( n \)-vertex graph with \( \kappa(G) > m^2 \), then \( \tau_R(G) \leq 8n^{5/4} \cdot n^{3/4} \).

As it is not known whether there exists a connected graph \( G \) with \( \text{lpt}(G) > 3 \), reducing the gap between our sublinear upper bound on \( \text{lpt}(G) \) and the constant lower bound remains a major open problem in the area of longest path transversals.
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References


