# Sublinear Longest Path Transversals 

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February 15, 2023


#### Abstract

We show that connected graphs admit sublinear longest path transversals. This improves an earlier result of Rautenbach and Sereni and is related to the fifty-year-old question of whether connected graphs admit longest path transversals of constant size. The same technique allows us to show that 2-connected graphs admit sublinear longest cycle transversals.


## 1 Introduction

A classical exercise in graph theory is to show that if $P$ and $Q$ are longest paths in a connected graph, then the vertex sets of $P$ and $Q$ have non-empty intersection (see [8], exercise 1.2.40). In 1966, Gallai [2] asked whether this result could be strengthened to assert that the family of all longest paths in a connected graph $G$ has non-empty intersection. It turns out the answer is no, as shown by Walther [6] with a 25 -vertex counterexample. A 12-vertex counterexample, due to Walther and Voss [7] and independently Zamfirescu [10], is obtained from the Petersen graph by replacing one vertex $v$ with an independent set $\left\{v_{1}, v_{2}, v_{3}\right\}$ such that each $v_{i}$ becomes an endpoint of an edge incident to $v$ (see Figure 1).

Since Gallai's question has a negative answer, a single vertex is generally insufficient to meet every longest path in a connected graph $G$. A longest path transversal in $G$ is a set of vertices that intersects every longest path. Such a set is a transversal in the hypergraph on $V(G)$ whose edges are the vertex sets of longest paths in $G$. Let $\operatorname{lpt}(G)$ be the minimum size of a longest path transversal in $G$. The graph $G_{0}$ in Figure 1 is a connected 12-vertex graph with $\operatorname{lpt}\left(G_{0}\right)=2$. Grünbaum [3] constructed a connected 324-vertex graph $G$ with $\operatorname{lpt}(G)=3$. Soon afterward, Zamfirescu [10] found such a graph with 270 vertices. Walther [6] and Zamfirescu [9] asked if $\operatorname{lpt}(G)$ is bounded for connected graphs $G$, and this remains

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Figure 1: The graph $G_{0}$ : a 12-vertex graph with $\operatorname{lpt}\left(G_{0}\right)=2$.
open. In fact, it is not known whether there is a connected graph $G$ with $\operatorname{lpt}(G) \geq 4$. Let $G$ be a connected graph. Since a connected graph does not contain vertex-disjoint longest paths, every partition of $V(G)$ into two sets has a part that contains no longest path in $G$, forcing the other part to be a longest path transversal. Applying this to a partition of $V(G)$ into two parts of nearly equal size gives $\operatorname{lpt}(G) \leq\lceil n / 2\rceil$ when $G$ is an $n$-vertex connected graph. It is not too difficult to improve this argument to obtain $\operatorname{lpt}(G) \leq\lceil n / 4\rceil$. Rautenbach and Sereni [4] showed that $\operatorname{lpt}(G) \leq\left\lceil\frac{n}{4}-\frac{n^{2 / 3}}{90}\right\rceil$ for every connected $n$-vertex graph $G$. We show that $\operatorname{lpt}(G) \leq 8 n^{3 / 4}$ when $G$ is an $n$-vertex connected graph, implying that connected graphs have sublinear longest path transversals.

Let $\operatorname{lct}(G)$ be the minimum size of a set of vertices $S$ such that $S$ intersects every longest cycle in $G$. Analogously to the case of longest paths in 1-connected graphs, every pair of longest cycles in a 2 -connected graph intersect. The Petersen graph $G$ is 2 -connected and $\operatorname{lct}(G)=2$. With no connectivity assumptions, Thomassen [5] showed that $\operatorname{lct}(G) \leq\lceil n / 3\rceil$ for each $n$-vertex graph $G$. The bound is sharp when $G$ is a disjoint union of triangles and nearly sharp in the 1-connected case when $G$ is obtained from a star with $(n-1) / 3$ leaves by replacing each leaf with a triangle. On the other hand, Rautenbach and Sereni [4] proved that if $G$ is 2 -connected, then $\operatorname{lct}(G) \leq\left\lceil\frac{n}{3}-\frac{n^{2 / 3}}{36}\right\rceil$. We show that $\operatorname{lct}(G) \leq 20 n^{3 / 4}$ when $G$ is 2-connected (Corollary 2).

The problems of finding small longest path transversals and small longest cycle transversals are special cases of a general problem that we aim to address. Given a multigraph $F$ and an edge $e \in E(F)$ with endpoints $u$ and $v$, the subdivision operation produces a new multigraph $F^{\prime}$ in which $e$ is replaced by a path uwv through a new vertex $w$ in $F^{\prime}$. A subdivision of $F$ is a graph obtained from $F$ via a sequence of zero or more subdivision operations. For a multigraph $R$ and a graph $G$, an $R$-subdivision in $G$ is a subgraph of $G$ isomorphic to a subdivision of $R$. We ask for a small set of vertices in $G$ that intersects every $R$-subdivision in $G$ of maximum size. The cases of longest path transversals and longest cycle transversals arise as $R=P_{2}$ and $R=C_{2}$ (the multigraph 2-vertex cycle), respectively. We prove that for each connected multigraph $R$, if the family $\mathcal{F}$ of maximum $R$-subdivisions in $G$ is pairwise intersecting, then $\mathcal{F}$ admits a transversal of size at most $C n^{3 / 4}$, where $C$ is a constant depending on $R$.

## 2 Maximum subdivision transversals

Let $R$ be a multigraph. Recall that an $R$-subdivision in $G$ is a subgraph of $G$ isomorphic to a subdivision of $R$, and a maximum $R$-subdivision is an $R$-subdivision $F$ in $G$ that maximizes $|V(F)|$. An $R$-transversal of $G$ is a set of vertices intersecting each maximum $R$-subdivision. Let $\tau_{R}(G)$ be the minimum size of an $R$-transversal in $G$.

Given sets of vertices $X$ and $Y$ of $G$, an $(X, Y)$-separator is a set of vertices $S$ such that no path in $G-S$ has one endpoint in $X$ and the other endpoint in $Y$. We allow an $(X, Y)$-separator to contain vertices in $X$ and $Y$. An $(X, Y)$-connector is a collection of vertex-disjoint paths $\left\{P_{1}, \ldots, P_{k}\right\}$ such that each $P_{i}$ has one endpoint in $X$, the other endpoint in $Y$, and the interior vertices of $P_{i}$ are outside $X \cup Y$. A variant of Menger's Theorem asserts that the minimum size of an $(X, Y)$-separator equals the maximum size of an $(X, Y)$-connector (see, e.g., Theorem 3.3.1 in [1]).

Our next result shows that when the maximum $R$-subdivisions in a graph $G$ pairwise intersect, $G$ has sublinear $R$-transversals. We make no attempt to optimize the multiplicative constant 8 or the dependence on $m$.

Theorem 1. Let $R$ be a connected $m$-edge multigraph with $m \geq 1$ and let $G$ be an $n$-vertex graph. If the maximum $R$-subdivisions in $G$ pairwise intersect, then $\tau_{R}(G) \leq 8 m^{5 / 4} n^{3 / 4}$.

Proof. Let $m=|E(R)|$ and let $\varepsilon=2(m / n)^{1 / 4}$. We may assume that $m \leq n$, since otherwise we may take $V(G)$ as our $R$-transversal. Let $\mathcal{F}$ be the family of maximum $R$-subdivisions in $G$. An $\varepsilon$-partial transversal is a triple $(H, X, Y)$ such that $H$ is a subgraph of $G, X=V(G)-$ $V(H), Y \subseteq X$ with $|Y| \leq \varepsilon|X|$, and each $F \in \mathcal{F}$ is a subgraph of $H$ or contains a vertex in $Y$. Given an $\varepsilon$-partial transversal $(H, X, Y)$, we either obtain an $\varepsilon$-partial transversal $\left(H^{\prime}, X^{\prime}, Y^{\prime}\right)$ with $\left|V\left(H^{\prime}\right)\right|<|V(H)|$ or we produce an $R$-transversal with at most $8 m^{5 / 4} n^{3 / 4}$ vertices. Starting with $(H, X, Y)=(G, \varnothing, \varnothing)$ and iterating gives the result.

Let $(H, X, Y)$ be an $\varepsilon$-partial transversal, and let $\mathcal{F}_{0}$ be the set of $F \in \mathcal{F}$ such that $F$ is a subgraph of $H$. We may assume that $H$ contains vertex-disjoint paths $P_{1}$ and $P_{2}$ each of size $\lceil\varepsilon n\rceil$. Otherwise, every path in $H$ has size less than $2\lceil\varepsilon n\rceil$, and so each $F \in \mathcal{F}_{0}$ has at most $2 m\lceil\varepsilon n\rceil$ vertices. Since $\mathcal{F}_{0}$ is pairwise intersecting, we have that $V(F) \cup Y$ is an $R$-transversal for each $F \in \mathcal{F}_{0}$. It follows that $\tau_{R}(G) \leq|Y|+2 m\lceil\varepsilon n\rceil \leq \varepsilon n+2 m\lceil\varepsilon n\rceil \leq$ $(2 m+1) \varepsilon n+2 m \leq(2 m+2) \varepsilon n \leq 4 m \varepsilon n=8 m^{5 / 4} n^{3 / 4}$.

Suppose that $H$ has a $\left(V\left(P_{1}\right), V\left(P_{2}\right)\right)$-separator $S$ of size at most $\varepsilon^{2} n$. Since graphs in $\mathcal{F}_{0}$ are connected, each $F \in \mathcal{F}_{0}$ has a vertex in $S$ or is contained in some component of $H-S$. Also, since $\mathcal{F}_{0}$ is pairwise intersecting, at most one component $H^{\prime}$ of $H-S$ contains graphs in $\mathcal{F}_{0}$. Since $S$ is a separator, $H^{\prime}$ is disjoint from at least one of $\left\{P_{1}, P_{2}\right\}$. With $X^{\prime}=V(G)-V\left(H^{\prime}\right)$ and $Y^{\prime}=Y \cup S$, we have $\left|X^{\prime}\right|-|X| \geq \varepsilon n$ and $\left|Y^{\prime}\right|=|Y|+|S| \leq$ $\varepsilon|X|+\varepsilon^{2} n \leq \varepsilon|X|+\varepsilon\left(\left|X^{\prime}\right|-|X|\right) \leq \varepsilon\left|X^{\prime}\right|$. It follows that $\left(H^{\prime}, X^{\prime}, Y^{\prime}\right)$ is an $\varepsilon$-partial transversal. Also $\left|V\left(H^{\prime}\right)\right|<|V(H)|$ since $\left|X^{\prime}\right|>|X|$.

Otherwise, by Menger's Theorem, $H$ has a $\left(V\left(P_{1}\right), V\left(P_{2}\right)\right)$-connector $\mathcal{P}$ with $|\mathcal{P}| \geq \varepsilon^{2} n$. Let $\mathcal{P}^{\prime}$ be the set of paths in $\mathcal{P}$ of size at most $2 / \varepsilon^{2}$. Note that $\left|\mathcal{P}^{\prime}\right| \geq|\mathcal{P}| / 2$, or else $\mathcal{P}$ has at least $\left(\varepsilon^{2} n\right) / 2$ paths of size more than $2 / \varepsilon^{2}$, contradicting that the paths in $\mathcal{P}$ are disjoint. So we have $\left|\mathcal{P}^{\prime}\right| \geq|\mathcal{P}| / 2 \geq\left(\varepsilon^{2} / 2\right) n=2 m^{1 / 2} n^{1 / 2} \geq 2$. Combining $P_{1}$ with two paths in $\mathcal{P}^{\prime}$ whose endpoints in $V\left(P_{1}\right)$ are as far apart as possible and a segment of $P_{2}$ gives a cycle $C_{0}$ such that $\left(\varepsilon^{2} / 2\right) n \leq\left|V\left(C_{0}\right)\right| \leq 2\lceil\varepsilon n\rceil+4 / \varepsilon^{2}-4 \leq 2 \varepsilon n+4 / \varepsilon^{2}$, where the lower bound


Figure 2: $(V(C), V(F))$-connector case. The subpath $W$ of the cycle $C$ is dashed, and the cycle $D$ is displayed in bold.
counts vertices in $V\left(P_{1}\right) \cap V\left(C_{0}\right)$ and the upper bound counts at most $2\lceil\varepsilon n\rceil$ vertices in $\left(V\left(P_{1}\right) \cup V\left(P_{2}\right)\right) \cap V\left(C_{0}\right)$, at most $4 / \varepsilon^{2}$ vertices on the paths in $\mathcal{P}^{\prime}$ linking $P_{1}$ and $P_{2}$, and observing that the 4 endpoints of the linking paths are counted twice.

Let $C$ be a longest cycle in $H$ subject to $|V(C)| \leq 2 \varepsilon n+4 / \varepsilon^{2}$, let $\ell=|V(C)|$, and note that $\ell \geq\left|V\left(C_{0}\right)\right| \geq\left(\varepsilon^{2} / 2\right) n$. If $V(C)$ intersects each subgraph in $\mathcal{F}_{0}$, then $Y \cup V(C)$ witnesses $\tau_{R}(G) \leq|V(C)|+|Y| \leq\left(2 \varepsilon n+4 / \varepsilon^{2}\right)+\varepsilon n=3 \varepsilon n+(n / m)^{1 / 2}<8 m^{5 / 4} n^{3 / 4}$. Otherwise, choose $F \in \mathcal{F}_{0}$ that is disjoint from $C$. We may assume $|V(F)| \geq \ell$, or else $Y \cup V(F)$ witnesses that $\tau_{R}(G) \leq|V(F)|+|Y|<\left(2 \varepsilon n+4 / \varepsilon^{2}\right)+\varepsilon n<8 m^{5 / 4} n^{3 / 4}$.

If $H$ has a $(V(C), V(F))$-separator $T$ of size at most $\varepsilon \ell$, then we obtain an $\varepsilon$-partial transversal as follows. At most one component $H^{\prime}$ of $H-T$ contains graphs in $\mathcal{F}_{0}$. Let $X^{\prime}=V(G)-V\left(H^{\prime}\right)$ and let $Y^{\prime}=Y \cup T$. Since $H^{\prime}$ is disjoint from one of $\{C, F\}$, it follows that $\left|X^{\prime}\right|-|X| \geq \ell$. We compute $\left|Y^{\prime}\right|=|Y|+|T| \leq \varepsilon|X|+\varepsilon \ell \leq \varepsilon|X|+\varepsilon\left(\left|X^{\prime}\right|-|X|\right) \leq \varepsilon\left|X^{\prime}\right|$. Hence ( $H^{\prime}, X^{\prime}, Y^{\prime}$ ) is an $\varepsilon$-partial transversal with $\left|V\left(H^{\prime}\right)\right|<|V(H)|$.

Otherwise, $H$ has a $(V(C), V(F))$-connector $\mathcal{Q}$ with $|\mathcal{Q}| \geq \varepsilon \ell$. We use $\mathcal{Q}$ to obtain a contradiction. For $e \in E(R)$, let $Q_{e}$ be the path in $F$ corresponding to $e$, and let $\mathcal{Q}_{e}$ be the set of paths in $\mathcal{Q}$ which have an endpoint in $Q_{e}$. Since $|E(R)|=m$, it follows that $\left|\mathcal{Q}_{e}\right| \geq|\mathcal{Q}| / m \geq \varepsilon \ell / m$ for some edge $e \in E(R)$. Let $\mathcal{Q}^{\prime}$ be the set of paths in $\mathcal{Q}_{e}$ of size at most $\frac{2 m n}{\varepsilon \ell}$, and note that $\left|\mathcal{Q}^{\prime}\right| \geq\left|\mathcal{Q}_{e}\right| / 2 \geq \frac{\varepsilon \ell}{2 m}$, or else $\mathcal{Q}_{e}$ has at least $\frac{\varepsilon \ell}{2 m}$ paths of size more than $\frac{2 m n}{\varepsilon \ell}$, a contradiction. The endpoints of paths in $\mathcal{Q}^{\prime}$ divide $Q_{e}$ into $\left|\mathcal{Q}^{\prime}\right|-1$ edge-disjoint subpaths. Choose $Q_{1}, Q_{2} \in \mathcal{Q}^{\prime}$ to minimize the length of such a subpath $Q_{0}$ of $Q_{e}$, and note that $Q_{0}$ has length at most $\frac{n-1}{\left|\mathcal{Q}^{\prime}\right|-1}$; see Figure 2. Since $m \leq n$, we have $2 m \leq 2 m^{3 / 4} n^{1 / 4}=\frac{\varepsilon^{3}}{4} n \leq \frac{\varepsilon \ell}{2}$, and hence $\frac{n-1}{\left|\mathcal{Q}^{\prime}\right|-1}<\frac{n}{\frac{\varepsilon \ell}{2 m}-1}=\frac{2 m n}{\varepsilon \ell-2 m} \leq \frac{4 m n}{\varepsilon \ell}$.

The endpoints of $Q_{1}$ and $Q_{2}$ on $C$ partition $C$ into two subpaths; let $W$ be the longer subpath. If $|E(W)| \geq\left|E\left(Q_{0}\right)\right|$, then we would obtain a larger $R$-subdivision by using $Q_{1}$, $W$, and $Q_{2}$ to bypass $Q_{0}$. Since $F$ is a maximum $R$-subdivision, we have $|E(W)|<\left|E\left(Q_{0}\right)\right|$. Therefore using $Q_{1}, Q_{0}$, and $Q_{2}$ to bypass $W$ gives a cycle $D$ with $|E(D)|>|E(C)|$. By the extremal choice of $C$, it follows that $|V(D)|>2 \varepsilon n+4 / \varepsilon^{2}$. On the other hand, $|V(D)|=$
$|E(D)| \leq \frac{\ell}{2}+\left|E\left(Q_{1}\right)\right|+\left|E\left(Q_{0}\right)\right|+\left|E\left(Q_{2}\right)\right| \leq \frac{\ell}{2}+\frac{2 m n}{\varepsilon \ell}+\frac{4 m n}{\varepsilon \ell}+\frac{2 m n}{\varepsilon \ell}=\frac{\ell}{2}+\frac{8 m n}{\varepsilon \ell}$.
Therefore $2 \varepsilon n+\frac{4}{\varepsilon^{2}}<|V(D)| \leq \frac{\ell}{2}+\frac{8 m n}{\varepsilon \ell} \leq \varepsilon n+\frac{2}{\varepsilon^{2}}+\frac{8 m n}{\varepsilon \ell} \leq \varepsilon n+\frac{2}{\varepsilon^{2}}+\frac{16 m}{\varepsilon^{3}}$, where the last inequality uses $\ell \geq\left(\varepsilon^{2} / 2\right) n$. Simplifying gives $\varepsilon n<\frac{16 m}{\varepsilon^{3}}-\frac{2}{\varepsilon^{2}}<\frac{16 m}{\varepsilon^{3}}$, and this inequality is violated when $\varepsilon \geq(16 m / n)^{1 / 4}$.

Applying Theorem 1, we obtain the following corollary.
Corollary 2. Let $G$ be an n-vertex graph. If $G$ is connected, then $\operatorname{lpt}(G) \leq 8 n^{3 / 4}$. If $G$ is 2 -connected, then $\operatorname{lct}(G) \leq 20 n^{3 / 4}$.

Proof. When $R=P_{2}$, an $R$-transversal is a longest path transversal. It is well known that if $G$ is connected, then the longest paths pairwise intersect. By Theorem 1, we have $\operatorname{lpt}(G)=\tau_{R}(G) \leq 8 n^{3 / 4}$.

Similarly, when $R=C_{2}$, an $R$-transversal is a longest cycle transversal. If $G$ is 2 connected, then the longest cycles pairwise intersect. By Theorem 1, we have $\operatorname{lct}(G)=$ $\tau_{R}(G) \leq 8 \cdot 2^{5 / 4} \cdot n^{3 / 4} \leq 20 n^{3 / 4}$.

We do not know whether the assumption in Theorem 1 that $R$ is connected is necessary to obtain sublinear $R$-transversals. To obtain analogues of Corollary 2 for general $R$, we show that the maximum $R$-subdivisions pairwise intersect when the connectivity of $G$ is sufficiently large. Recall that a graph $G$ is $k$-connected if $|V(G)|>k$ and $G-S$ is connected for each $S \subseteq V(G)$ with $|S|<k$. Moreover, the connectivity of $G$, denoted $\kappa(G)$, is the maximum $k$ such that G is $k$-connected.

Lemma 3. Let $R$ be a connected $m$-edge multigraph with $m \geq 1$. If $\kappa(G)>m^{2}$, then the maximum $R$-subdivisions in $G$ are pairwise intersecting.

Proof. Suppose for a contradiction that $G$ has disjoint maximum $R$-subdivisions $F_{1}$ and $F_{2}$, and let $k=\left|V\left(F_{1}\right)\right|=\left|V\left(F_{2}\right)\right|$. By Menger's Theorem, there is an $\left(V\left(F_{1}\right), V\left(F_{2}\right)\right)$-connector $\mathcal{P}$ with $|\mathcal{P}|=\min \left\{k, m^{2}+1\right\}$. If $|\mathcal{P}|=k$, then every vertex in $F_{1}$ is an endpoint of a path in $\mathcal{P}$, and we obtain an $R$-subdivision of size more than $k$ by replacing an edge $u v \in E\left(F_{1}\right)$ with a path in $\mathcal{P}$ having $u$ as an endpoint, a path in $\mathcal{P}$ having $v$ as an endpoint, and an appropriate path in the connected subgraph $F_{2}$.

So we may assume $|\mathcal{P}|=m^{2}+1$. For each $e \in E(R)$, let $F_{i}(e)$ be the path in $F_{i}$ corresponding to $e$. Since $R$ has no isolated vertices, we may associate each $P \in \mathcal{P}$ with an ordered pair of edges $\left(e_{1}, e_{2}\right) \in(E(R))^{2}$ such that $P$ has its endpoint in $F_{1}$ in $F_{1}\left(e_{1}\right)$ and its endpoint in $F_{2}$ in $F_{2}\left(e_{2}\right)$. Since $|\mathcal{P}|>m^{2}$, some pair $\left(e_{1}, e_{2}\right)$ is associated with distinct paths $P, Q \in \mathcal{P}$. Let $W_{i}$ be the subpath of $F_{i}\left(e_{i}\right)$ whose endpoints are in $V(P) \cup V(Q)$. If $\left|E\left(W_{1}\right)\right| \geq\left|E\left(W_{2}\right)\right|$, then we modify $F_{2}$ to obtain a larger $R$-subdivision by using $P, W_{1}$, and $Q$ to bypass $W_{2}$. Similarly, if $\left|E\left(W_{2}\right)\right| \geq\left|E\left(W_{1}\right)\right|$, then we modify $F_{1}$ to obtain a larger $R$-subdivision by using $P, W_{2}$, and $Q$ to bypass $W_{1}$.

Corollary 4. Let $R$ be a connected m-edge multigraph. If $G$ is an n-vertex graph with $\kappa(G)>m^{2}$, then $\tau_{R}(G) \leq 8 m^{5 / 4} n^{3 / 4}$.

As it is not known whether there exists a connected graph $G$ with $\operatorname{lpt}(G)>3$, reducing the gap between our sublinear upper bound on $\operatorname{lpt}(G)$ and the constant lower bound remains a major open problem in the area of longest path transversals.

## Acknowledgement

The authors greatly appreciate the careful comments of an anonymous referee.

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