Sublinear Longest Path Transversals

James A. Long Jr.^{*1}, Kevin G. Milans^{†1}, and Andrea Munaro^{‡2}

¹Department of Mathematics, West Virginia University, USA ²School of Mathematics and Physics, Queen's University Belfast, UK

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Abstract

We show that connected graphs admit sublinear longest path transversals. This improves an earlier result of Rautenbach and Sereni and is related to the fifty-year-old question of whether connected graphs admit longest path transversals of constant size. The same technique allows us to show that 2-connected graphs admit sublinear longest cycle transversals.

1 Introduction

A classical exercise in graph theory is to show that if P and Q are longest paths in a connected graph, then the vertex sets of P and Q have non-empty intersection (see [8], exercise 1.2.40). In 1966, Gallai [2] asked whether this result could be strengthened to assert that the family of all longest paths in a connected graph G has non-empty intersection. It turns out the answer is no, as shown by Walther [6] with a 25-vertex counterexample. A 12-vertex counterexample, due to Walther and Voss [7] and independently Zamfirescu [10], is obtained from the Petersen graph by replacing one vertex v with an independent set $\{v_1, v_2, v_3\}$ such that each v_i becomes an endpoint of an edge incident to v (see Figure 1).

Since Gallai's question has a negative answer, a single vertex is generally insufficient to meet every longest path in a connected graph G. A longest path transversal in G is a set of vertices that intersects every longest path. Such a set is a transversal in the hypergraph on V(G) whose edges are the vertex sets of longest paths in G. Let lpt(G) be the minimum size of a longest path transversal in G. The graph G_0 in Figure 1 is a connected 12-vertex graph with $lpt(G_0) = 2$. Grünbaum [3] constructed a connected 324-vertex graph G with lpt(G) = 3. Soon afterward, Zamfirescu [10] found such a graph with 270 vertices. Walther [6] and Zamfirescu [9] asked if lpt(G) is bounded for connected graphs G, and this remains

^{*}jalong@mix.wvu.edu

[†]milans@math.wvu.edu

[‡]a.munaro@qub.ac.uk

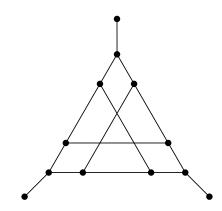


Figure 1: The graph G_0 : a 12-vertex graph with $lpt(G_0) = 2$.

open. In fact, it is not known whether there is a connected graph G with $lpt(G) \ge 4$. Let G be a connected graph. Since a connected graph does not contain vertex-disjoint longest paths, every partition of V(G) into two sets has a part that contains no longest path in G, forcing the other part to be a longest path transversal. Applying this to a partition of V(G) into two parts of nearly equal size gives $lpt(G) \le \lceil n/2 \rceil$ when G is an n-vertex connected graph. It is not too difficult to improve this argument to obtain $lpt(G) \le \lceil n/4 \rceil$. Rautenbach and Sereni [4] showed that $lpt(G) \le \lceil \frac{n}{4} - \frac{n^{2/3}}{90} \rceil$ for every connected n-vertex graph G. We show that $lpt(G) \le 8n^{3/4}$ when G is an n-vertex connected graph, implying that connected graphs have sublinear longest path transversals.

Let $\operatorname{lct}(G)$ be the minimum size of a set of vertices S such that S intersects every longest cycle in G. Analogously to the case of longest paths in 1-connected graphs, every pair of longest cycles in a 2-connected graph intersect. The Petersen graph G is 2-connected and $\operatorname{lct}(G) = 2$. With no connectivity assumptions, Thomassen [5] showed that $\operatorname{lct}(G) \leq \lceil n/3 \rceil$ for each *n*-vertex graph G. The bound is sharp when G is a disjoint union of triangles and nearly sharp in the 1-connected case when G is obtained from a star with (n-1)/3 leaves by replacing each leaf with a triangle. On the other hand, Rautenbach and Sereni [4] proved that if G is 2-connected, then $\operatorname{lct}(G) \leq \lceil \frac{n}{3} - \frac{n^{2/3}}{36} \rceil$. We show that $\operatorname{lct}(G) \leq 20n^{3/4}$ when G is 2-connected (Corollary 2).

The problems of finding small longest path transversals and small longest cycle transversals are special cases of a general problem that we aim to address. Given a multigraph F and an edge $e \in E(F)$ with endpoints u and v, the subdivision operation produces a new multigraph F' in which e is replaced by a path uwv through a new vertex w in F'. A subdivision of F is a graph obtained from F via a sequence of zero or more subdivision operations. For a multigraph R and a graph G, an R-subdivision in G is a subgraph of G isomorphic to a subdivision of R. We ask for a small set of vertices in G that intersects every R-subdivision in G of maximum size. The cases of longest path transversals and longest cycle transversals arise as $R = P_2$ and $R = C_2$ (the multigraph 2-vertex cycle), respectively. We prove that for each connected multigraph R, if the family \mathcal{F} of maximum R-subdivisions in G is a constant depending on R.

2 Maximum subdivision transversals

Let R be a multigraph. Recall that an R-subdivision in G is a subgraph of G isomorphic to a subdivision of R, and a maximum R-subdivision is an R-subdivision F in G that maximizes |V(F)|. An R-transversal of G is a set of vertices intersecting each maximum R-subdivision. Let $\tau_R(G)$ be the minimum size of an R-transversal in G.

Given sets of vertices X and Y of G, an (X, Y)-separator is a set of vertices S such that no path in G - S has one endpoint in X and the other endpoint in Y. We allow an (X, Y)-separator to contain vertices in X and Y. An (X, Y)-connector is a collection of vertex-disjoint paths $\{P_1, \ldots, P_k\}$ such that each P_i has one endpoint in X, the other endpoint in Y, and the interior vertices of P_i are outside $X \cup Y$. A variant of Menger's Theorem asserts that the minimum size of an (X, Y)-separator equals the maximum size of an (X, Y)-connector (see, e.g., Theorem 3.3.1 in [1]).

Our next result shows that when the maximum R-subdivisions in a graph G pairwise intersect, G has sublinear R-transversals. We make no attempt to optimize the multiplicative constant 8 or the dependence on m.

Theorem 1. Let R be a connected m-edge multigraph with $m \ge 1$ and let G be an n-vertex graph. If the maximum R-subdivisions in G pairwise intersect, then $\tau_R(G) \le 8m^{5/4}n^{3/4}$.

Proof. Let m = |E(R)| and let $\varepsilon = 2(m/n)^{1/4}$. We may assume that $m \leq n$, since otherwise we may take V(G) as our *R*-transversal. Let \mathcal{F} be the family of maximum *R*-subdivisions in *G*. An ε -partial transversal is a triple (H, X, Y) such that *H* is a subgraph of *G*, X = V(G) - V(H), $Y \subseteq X$ with $|Y| \leq \varepsilon |X|$, and each $F \in \mathcal{F}$ is a subgraph of *H* or contains a vertex in *Y*. Given an ε -partial transversal (H, X, Y), we either obtain an ε -partial transversal (H', X', Y') with |V(H')| < |V(H)| or we produce an *R*-transversal with at most $8m^{5/4}n^{3/4}$ vertices. Starting with $(H, X, Y) = (G, \emptyset, \emptyset)$ and iterating gives the result.

Let (H, X, Y) be an ε -partial transversal, and let \mathcal{F}_0 be the set of $F \in \mathcal{F}$ such that Fis a subgraph of H. We may assume that H contains vertex-disjoint paths P_1 and P_2 each of size $\lceil \varepsilon n \rceil$. Otherwise, every path in H has size less than $2 \lceil \varepsilon n \rceil$, and so each $F \in \mathcal{F}_0$ has at most $2m \lceil \varepsilon n \rceil$ vertices. Since \mathcal{F}_0 is pairwise intersecting, we have that $V(F) \cup Y$ is an R-transversal for each $F \in \mathcal{F}_0$. It follows that $\tau_R(G) \leq |Y| + 2m \lceil \varepsilon n \rceil \leq \varepsilon n + 2m \lceil \varepsilon n \rceil \leq$ $(2m+1)\varepsilon n + 2m \leq (2m+2)\varepsilon n \leq 4m\varepsilon n = 8m^{5/4}n^{3/4}$.

Suppose that H has a $(V(P_1), V(P_2))$ -separator S of size at most $\varepsilon^2 n$. Since graphs in \mathcal{F}_0 are connected, each $F \in \mathcal{F}_0$ has a vertex in S or is contained in some component of H-S. Also, since \mathcal{F}_0 is pairwise intersecting, at most one component H' of H-S contains graphs in \mathcal{F}_0 . Since S is a separator, H' is disjoint from at least one of $\{P_1, P_2\}$. With X' = V(G) - V(H') and $Y' = Y \cup S$, we have $|X'| - |X| \ge \varepsilon n$ and $|Y'| = |Y| + |S| \le \varepsilon |X| + \varepsilon^2 n \le \varepsilon |X| + \varepsilon (|X'| - |X|) \le \varepsilon |X'|$. It follows that (H', X', Y') is an ε -partial transversal. Also |V(H')| < |V(H)| since |X'| > |X|.

Otherwise, by Menger's Theorem, H has a $(V(P_1), V(P_2))$ -connector \mathcal{P} with $|\mathcal{P}| \geq \varepsilon^2 n$. Let \mathcal{P}' be the set of paths in \mathcal{P} of size at most $2/\varepsilon^2$. Note that $|\mathcal{P}'| \geq |\mathcal{P}|/2$, or else \mathcal{P} has at least $(\varepsilon^2 n)/2$ paths of size more than $2/\varepsilon^2$, contradicting that the paths in \mathcal{P} are disjoint. So we have $|\mathcal{P}'| \geq |\mathcal{P}|/2 \geq (\varepsilon^2/2)n = 2m^{1/2}n^{1/2} \geq 2$. Combining P_1 with two paths in \mathcal{P}' whose endpoints in $V(P_1)$ are as far apart as possible and a segment of P_2 gives a cycle C_0 such that $(\varepsilon^2/2)n \leq |V(C_0)| \leq 2 \lceil \varepsilon n \rceil + 4/\varepsilon^2 - 4 \leq 2\varepsilon n + 4/\varepsilon^2$, where the lower bound

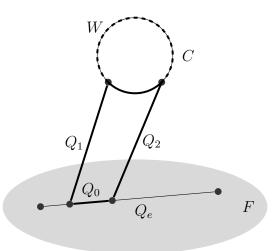


Figure 2: (V(C), V(F))-connector case. The subpath W of the cycle C is dashed, and the cycle D is displayed in bold.

counts vertices in $V(P_1) \cap V(C_0)$ and the upper bound counts at most $2 \lceil \varepsilon n \rceil$ vertices in $(V(P_1) \cup V(P_2)) \cap V(C_0)$, at most $4/\varepsilon^2$ vertices on the paths in \mathcal{P}' linking P_1 and P_2 , and observing that the 4 endpoints of the linking paths are counted twice.

Let C be a longest cycle in H subject to $|V(C)| \leq 2\varepsilon n + 4/\varepsilon^2$, let $\ell = |V(C)|$, and note that $\ell \geq |V(C_0)| \geq (\varepsilon^2/2)n$. If V(C) intersects each subgraph in \mathcal{F}_0 , then $Y \cup V(C)$ witnesses $\tau_R(G) \leq |V(C)| + |Y| \leq (2\varepsilon n + 4/\varepsilon^2) + \varepsilon n = 3\varepsilon n + (n/m)^{1/2} < 8m^{5/4}n^{3/4}$. Otherwise, choose $F \in \mathcal{F}_0$ that is disjoint from C. We may assume $|V(F)| \geq \ell$, or else $Y \cup V(F)$ witnesses that $\tau_R(G) \leq |V(F)| + |Y| < (2\varepsilon n + 4/\varepsilon^2) + \varepsilon n < 8m^{5/4}n^{3/4}$.

If H has a (V(C), V(F))-separator T of size at most $\varepsilon \ell$, then we obtain an ε -partial transversal as follows. At most one component H' of H - T contains graphs in \mathcal{F}_0 . Let X' = V(G) - V(H') and let $Y' = Y \cup T$. Since H' is disjoint from one of $\{C, F\}$, it follows that $|X'| - |X| \ge \ell$. We compute $|Y'| = |Y| + |T| \le \varepsilon |X| + \varepsilon \ell \le \varepsilon |X| + \varepsilon (|X'| - |X|) \le \varepsilon |X'|$. Hence (H', X', Y') is an ε -partial transversal with |V(H')| < |V(H)|.

Otherwise, H has a (V(C), V(F))-connector \mathcal{Q} with $|\mathcal{Q}| \geq \varepsilon \ell$. We use \mathcal{Q} to obtain a contradiction. For $e \in E(R)$, let Q_e be the path in F corresponding to e, and let \mathcal{Q}_e be the set of paths in \mathcal{Q} which have an endpoint in Q_e . Since |E(R)| = m, it follows that $|\mathcal{Q}_e| \geq |\mathcal{Q}|/m \geq \varepsilon \ell/m$ for some edge $e \in E(R)$. Let \mathcal{Q}' be the set of paths in \mathcal{Q}_e of size at most $\frac{2mn}{\varepsilon \ell}$, and note that $|\mathcal{Q}'| \geq |\mathcal{Q}_e|/2 \geq \frac{\varepsilon \ell}{2m}$, or else \mathcal{Q}_e has at least $\frac{\varepsilon \ell}{2m}$ paths of size more than $\frac{2mn}{\varepsilon \ell}$, a contradiction. The endpoints of paths in \mathcal{Q}' divide Q_e into $|\mathcal{Q}'| - 1$ edge-disjoint subpaths. Choose $Q_1, Q_2 \in \mathcal{Q}'$ to minimize the length of such a subpath Q_0 of Q_e , and note that Q_0 has length at most $\frac{n-1}{|\mathcal{Q}'|-1}$; see Figure 2. Since $m \leq n$, we have $2m \leq 2m^{3/4}n^{1/4} = \frac{\varepsilon^3}{4}n \leq \frac{\varepsilon \ell}{2}$, and hence $\frac{n-1}{|\mathcal{Q}'|-1} < \frac{n}{\varepsilon \ell} = \frac{2mn}{\varepsilon \ell - 2m} \leq \frac{4mn}{\varepsilon \ell}$.

The endpoints of Q_1 and Q_2 on C partition \tilde{C} into two subpaths; let W be the longer subpath. If $|E(W)| \ge |E(Q_0)|$, then we would obtain a larger R-subdivision by using Q_1 , W, and Q_2 to bypass Q_0 . Since F is a maximum R-subdivision, we have $|E(W)| < |E(Q_0)|$. Therefore using Q_1 , Q_0 , and Q_2 to bypass W gives a cycle D with |E(D)| > |E(C)|. By the extremal choice of C, it follows that $|V(D)| > 2\varepsilon n + 4/\varepsilon^2$. On the other hand, |V(D)| =

$$\begin{split} |E(D)| &\leq \frac{\ell}{2} + |E(Q_1)| + |E(Q_0)| + |E(Q_2)| \leq \frac{\ell}{2} + \frac{2mn}{\varepsilon\ell} + \frac{4mn}{\varepsilon\ell} + \frac{2mn}{\varepsilon\ell} = \frac{\ell}{2} + \frac{8mn}{\varepsilon\ell}.\\ \text{Therefore } 2\varepsilon n + \frac{4}{\varepsilon^2} < |V(D)| \leq \frac{\ell}{2} + \frac{8mn}{\varepsilon\ell} \leq \varepsilon n + \frac{2}{\varepsilon^2} + \frac{8mn}{\varepsilon\ell} \leq \varepsilon n + \frac{2}{\varepsilon^2} + \frac{16m}{\varepsilon^3}, \text{ where the last inequality uses } \ell \geq (\varepsilon^2/2)n. \text{ Simplifying gives } \varepsilon n < \frac{16m}{\varepsilon^3} - \frac{2}{\varepsilon^2} < \frac{16m}{\varepsilon^3}, \text{ and this inequality is } \varepsilon n < \frac{16m}{\varepsilon^3} - \frac{2}{\varepsilon^2} < \frac{16m}{\varepsilon^3}, \text{ and this inequality is } \varepsilon n < \frac{16m}{\varepsilon^3} - \frac{2}{\varepsilon^2} < \frac{16m}{\varepsilon^3}, \text{ and this inequality is } \varepsilon n < \frac{16m}{\varepsilon^3} - \frac{2}{\varepsilon^2} < \frac{16m}{\varepsilon^3}, \text{ and this inequality is } \varepsilon n < \frac{16m}{\varepsilon^3} - \frac{2}{\varepsilon^2} < \frac{16m}{\varepsilon^3}, \text{ and this inequality is } \varepsilon n < \frac{16m}{\varepsilon^3} - \frac{2}{\varepsilon^2} < \frac{16m}{\varepsilon^3} - \frac{2}{\varepsilon^3} < \frac{16m}{\varepsilon^3} - \frac{16m}{\varepsilon^3} - \frac{2}{\varepsilon^3} < \frac{16m}{\varepsilon^3} - \frac{2}{\varepsilon^3} < \frac{16m}{\varepsilon^3} - \frac{2}{\varepsilon^3} < \frac{16m}{\varepsilon^3} - \frac{2}{\varepsilon^3} < \frac{16m}{\varepsilon^3} - \frac{16m}{\varepsilon^3} - \frac{16m}{\varepsilon^3} - \frac{16m}{\varepsilon^3} - \frac{16m}{\varepsilon^3} - \frac{16m}{\varepsilon^3}$$
violated when $\varepsilon > (16m/n)^{1/4}$.

Applying Theorem 1, we obtain the following corollary.

Corollary 2. Let G be an n-vertex graph. If G is connected, then $lpt(G) \leq 8n^{3/4}$. If G is 2-connected, then $lct(G) \leq 20n^{3/4}$.

Proof. When $R = P_2$, an R-transversal is a longest path transversal. It is well known that if G is connected, then the longest paths pairwise intersect. By Theorem 1, we have $lpt(G) = \tau_R(G) \le 8n^{3/4}.$

Similarly, when $R = C_2$, an R-transversal is a longest cycle transversal. If G is 2connected, then the longest cycles pairwise intersect. By Theorem 1, we have lct(G) = $\tau_R(G) \le 8 \cdot 2^{5/4} \cdot n^{3/4} \le 20n^{3/4}.$

We do not know whether the assumption in Theorem 1 that R is connected is necessary to obtain sublinear R-transversals. To obtain analogues of Corollary 2 for general R, we show that the maximum R-subdivisions pairwise intersect when the connectivity of G is sufficiently large. Recall that a graph G is k-connected if |V(G)| > k and G - S is connected for each $S \subseteq V(G)$ with |S| < k. Moreover, the *connectivity* of G, denoted $\kappa(G)$, is the maximum k such that G is k-connected.

Lemma 3. Let R be a connected m-edge multigraph with $m \ge 1$. If $\kappa(G) > m^2$, then the maximum R-subdivisions in G are pairwise intersecting.

Proof. Suppose for a contradiction that G has disjoint maximum R-subdivisions F_1 and F_2 , and let $k = |V(F_1)| = |V(F_2)|$. By Menger's Theorem, there is an $(V(F_1), V(F_2))$ -connector \mathcal{P} with $|\mathcal{P}| = \min\{k, m^2 + 1\}$. If $|\mathcal{P}| = k$, then every vertex in F_1 is an endpoint of a path in \mathcal{P} , and we obtain an R-subdivision of size more than k by replacing an edge $uv \in E(F_1)$ with a path in \mathcal{P} having u as an endpoint, a path in \mathcal{P} having v as an endpoint, and an appropriate path in the connected subgraph F_2 .

So we may assume $|\mathcal{P}| = m^2 + 1$. For each $e \in E(R)$, let $F_i(e)$ be the path in F_i corresponding to e. Since R has no isolated vertices, we may associate each $P \in \mathcal{P}$ with an ordered pair of edges $(e_1, e_2) \in (E(R))^2$ such that P has its endpoint in F_1 in $F_1(e_1)$ and its endpoint in F_2 in $F_2(e_2)$. Since $|\mathcal{P}| > m^2$, some pair (e_1, e_2) is associated with distinct paths $P, Q \in \mathcal{P}$. Let W_i be the subpath of $F_i(e_i)$ whose endpoints are in $V(P) \cup V(Q)$. If $|E(W_1)| \geq |E(W_2)|$, then we modify F_2 to obtain a larger R-subdivision by using P, W_1 , and Q to bypass W_2 . Similarly, if $|E(W_2)| \geq |E(W_1)|$, then we modify F_1 to obtain a larger *R*-subdivision by using P, W_2 , and Q to bypass W_1 .

Corollary 4. Let R be a connected m-edge multigraph. If G is an n-vertex graph with $\kappa(G) > m^2$, then $\tau_R(G) \leq 8m^{5/4}n^{3/4}$.

As it is not known whether there exists a connected graph G with lpt(G) > 3, reducing the gap between our sublinear upper bound on lpt(G) and the constant lower bound remains a major open problem in the area of longest path transversals.

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