Longest Path Transversals and Gallai Families

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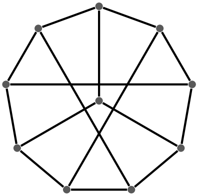

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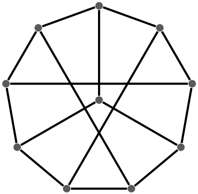

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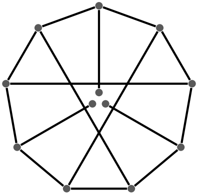

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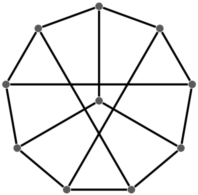

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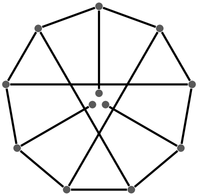

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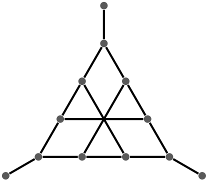

Redrawn

Two natural questions

- How many vertices are needed to hit every longest path in an n-vertex connected graph?

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- For which graphs does Gallai's question have a positive answer?

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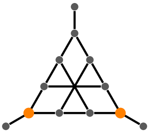

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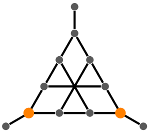

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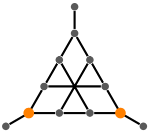

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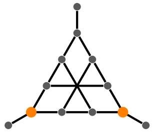

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- Let $\tau(n) = \max\{\tau(G) \colon |V(G)| = n \text{ and } G \text{ is connected}\}$.
- Probably $\tau(n)$ is small, perhaps even bounded.

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- Let $A, B, S \subseteq V(G)$. The set S is an (A, B)-separator if $G - S$ has no path from a vertex in A to a vertex in B.

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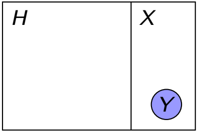
- Theorem (Menger): the min. size of an (A, B)-separator equals the max. size of an (A, B)-connector.

Partial Transversals



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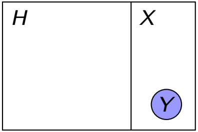

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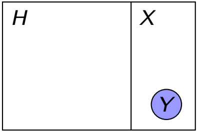

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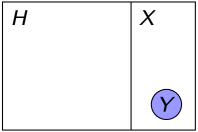

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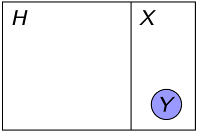

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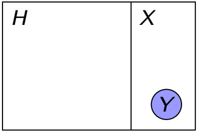

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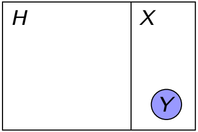

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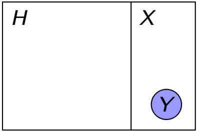

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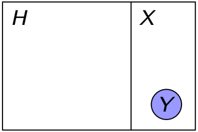

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- Start with $(H, X, Y) = (G, \varnothing, \varnothing)$ and iterate.

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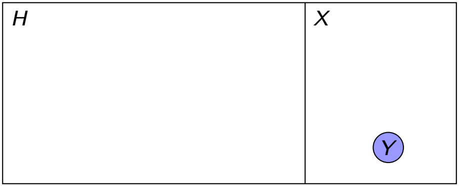

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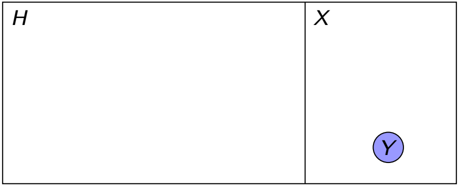

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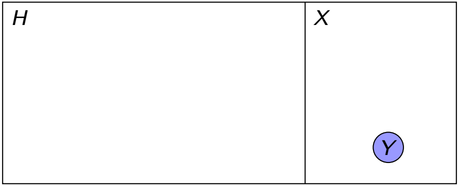

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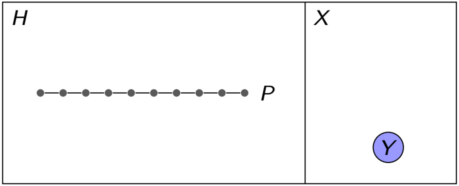

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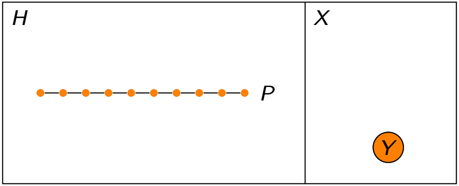

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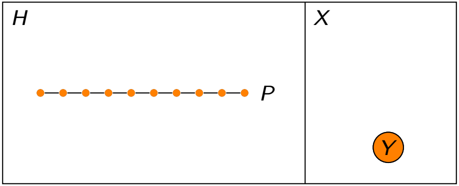

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  - $\tau(G) \leq \max\{|Y|, |V(P)|\} \leq 2\varepsilon n$.

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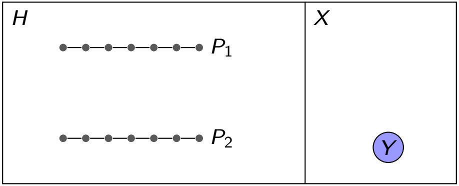

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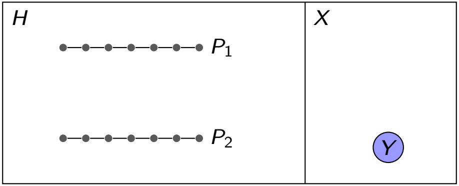

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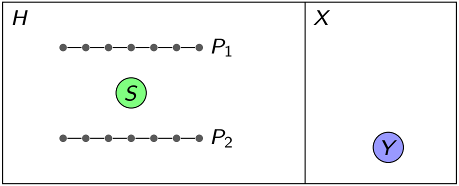

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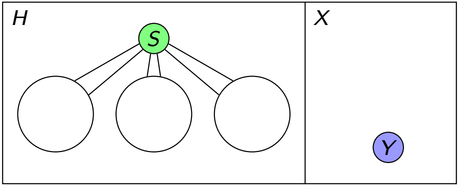

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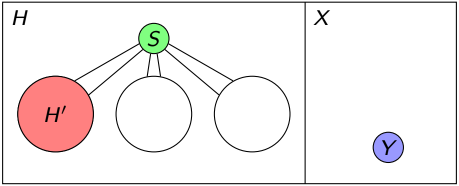

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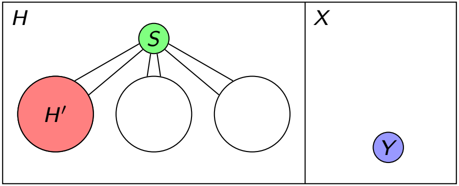

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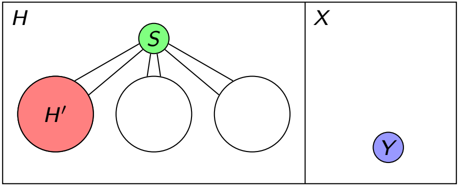

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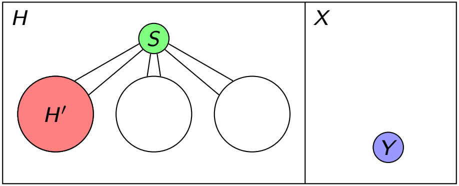

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  - Note that H' is disjoint from P_1 or P_2.

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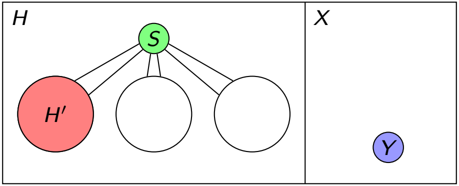

- Let P_1, P_2 be disjoint paths in H of size εn.
- H has a $(V(P_1), V(P_2))$-separator S with $|S| \leq \varepsilon^2 n$ or a $(V(P_1), V(P_2))$-connector $\mathcal{Q}$ with $|\mathcal{Q}| \geq \varepsilon^2 n$.
- If we get a separator S:
  - At most one component H' of $H - S$ contains paths in $\mathcal{L}(G)$.
  - Let $X' = X \cup (V(H) - V(H'))$.
  - Let $Y' = Y \cup S$.
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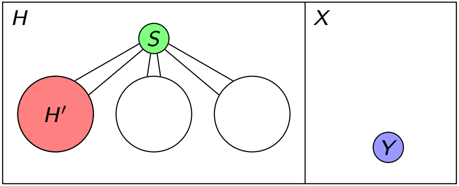
Partial Transversal Refinement



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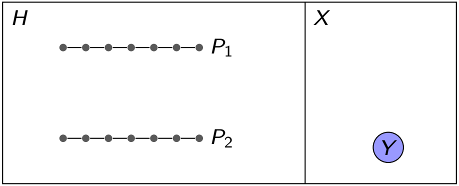
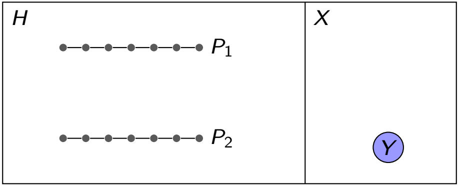
  - (H', X', Y') is an ε-partial transversal.

Partial Transversal Refinement



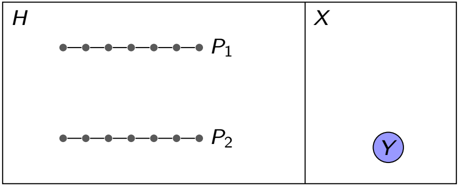
- Let $\mathcal{Q}$ be a $(V(P_1), V(P_2))$-connector with $|\mathcal{Q}| \geq \varepsilon^2 n$.

Partial Transversal Refinement



- Let $\mathcal{Q}$ be a $(V(P_1), V(P_2))$-connector with $|\mathcal{Q}| \geq \varepsilon^2 n$.
- Average size of a path in $\mathcal{Q}$ is at most $\frac{n}{\varepsilon^2 n}$, or $\frac{1}{\varepsilon^2}$.

Partial Transversal Refinement



- Let $\mathcal{Q}$ be a $(V(P_1), V(P_2))$-connector with $|\mathcal{Q}| \geq \varepsilon^2 n$.
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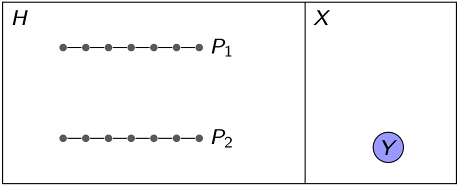
- At least half of the paths in $\mathcal{Q}$ have at most $\frac{2}{\varepsilon^2}$ vertices.

Partial Transversal Refinement



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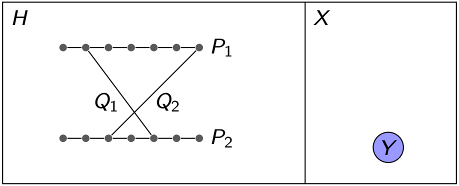
Partial Transversal Refinement



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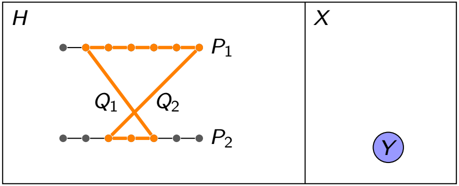
Partial Transversal Refinement



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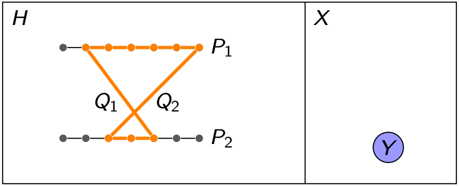
- Let C be the cycle formed by P_1, P_2, Q_1, and Q_2.

Partial Transversal Refinement



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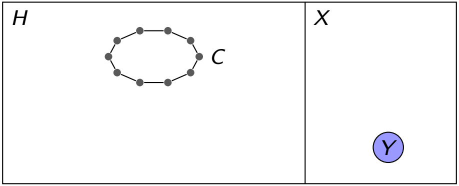
- Let C be the cycle formed by P_1, P_2, Q_1, and Q_2.
- Note that $\frac{\varepsilon^2 n}{2} \leq |V(C)| \leq 2\varepsilon n + 2 \cdot \frac{2}{\varepsilon^2}$.

Partial Transversal Refinement



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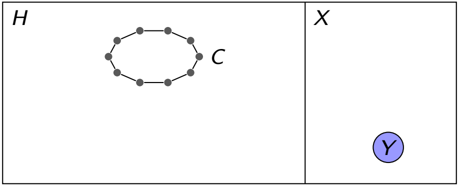
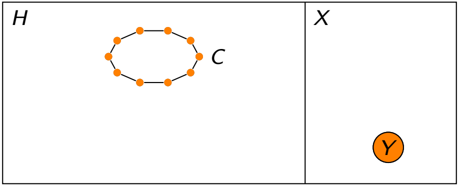
Partial Transversal Refinement



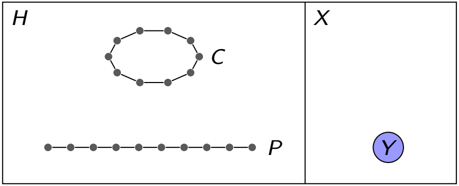
- $\frac{\varepsilon^2 n}{2} \leq |V(C)| \leq 2\varepsilon n + 2 \cdot \frac{2}{\varepsilon^2}$
- Is $V(C) \cup Y$ a Gallai set?

Partial Transversal Refinement



- $\frac{\varepsilon^2 n}{2} \leq |V(C)| \leq 2\varepsilon n + 2 \cdot \frac{2}{\varepsilon^2}$
- Is $V(C) \cup Y$ a Gallai set?
- If yes, then $\tau(G) \leq |Y| + |V(C)| \leq 3\varepsilon n + \frac{4}{\varepsilon^2}$.

Partial Transversal Refinement



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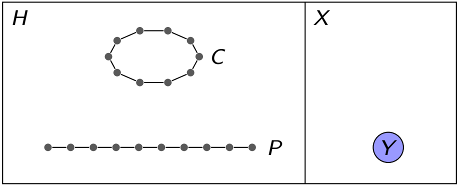
- Otherwise, H contains a path $P \in \mathcal{L}(G)$ disjoint from C.

Partial Transversal Refinement



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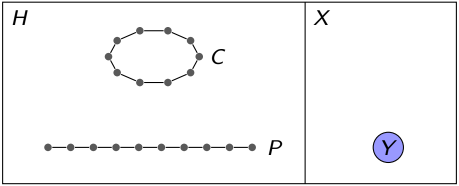
Partial Transversal Refinement



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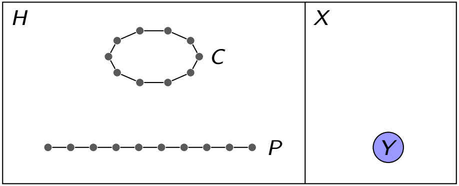

- $|V(P)| \geq |V(C)| \geq \frac{\varepsilon^2 n}{2}$.
- H has a $(V(C), V(P))$-separator S with $|S| \leq \frac{\varepsilon^3 n}{2}$ or a $(V(C), V(P))$-connector $\mathcal{Q}$ with $|\mathcal{Q}| \geq \frac{\varepsilon^3 n}{2}$.

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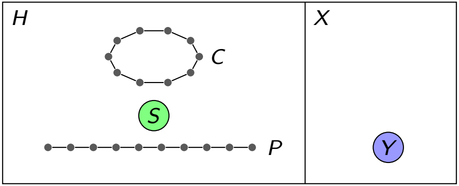

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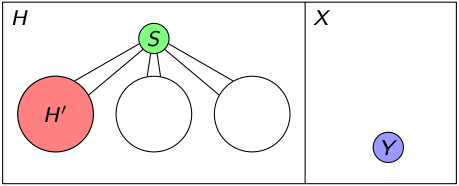

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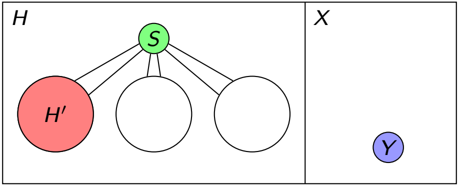

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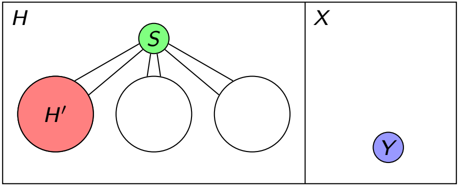

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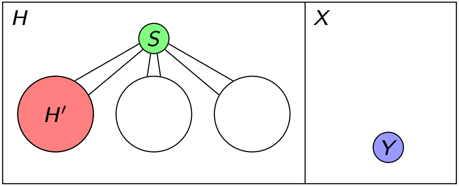

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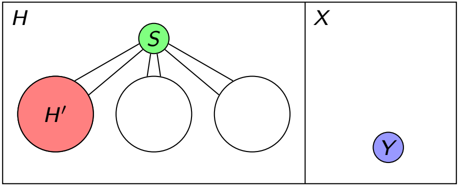

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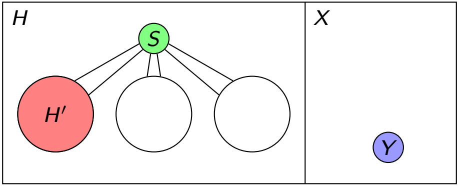

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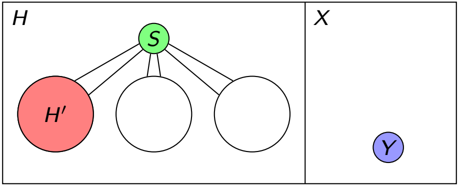
Partial Transversal Refinement



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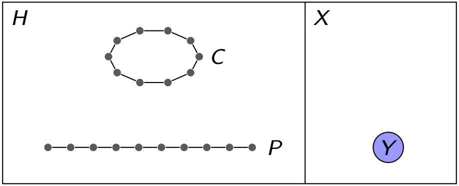
Partial Transversal Refinement



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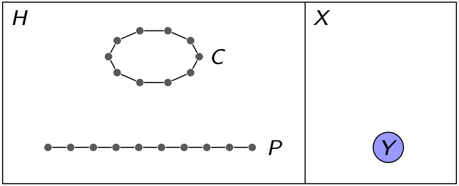
Partial Transversal Refinement



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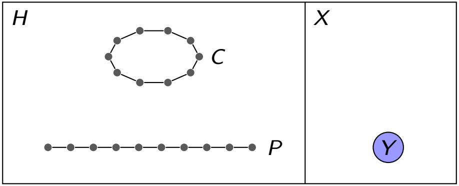

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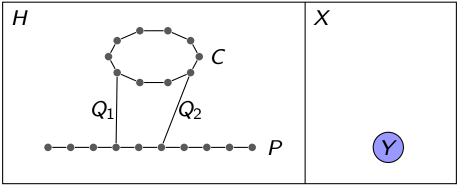
Partial Transversal Refinement



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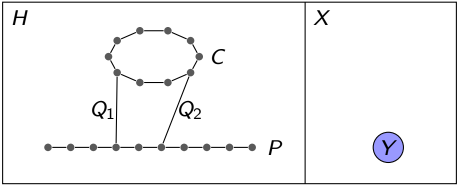

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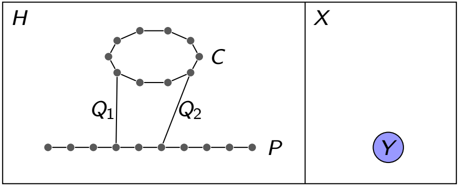
- Choose $Q_1, Q_2 \in \mathcal{Q}$ to minimize dist. between endpoints in P.
- Dist. between ends of Q_1 and Q_2 on P is roughly at most $\frac{2}{\varepsilon^3}$.

Partial Transversal Refinement



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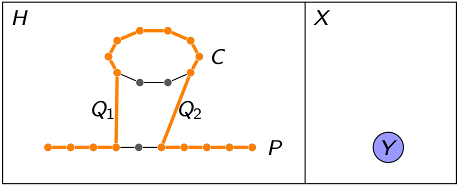
- Choose $Q_1, Q_2 \in \mathcal{Q}$ to minimize dist. between endpoints in P.
- Dist. between ends of Q_1 and Q_2 on P is roughly at most $\frac{2}{\varepsilon^3}$.
- If $\frac{2}{\varepsilon^3} < \frac{1}{2}|V(C)|$, then we obtain a longer path.

Partial Transversal Refinement



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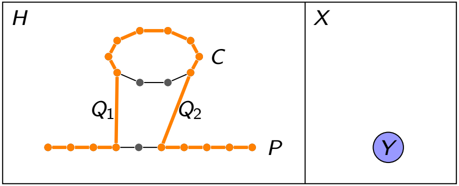

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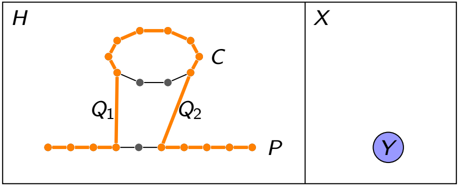
- Dist. between ends of Q_1 and Q_2 on P is roughly at most $\frac{2}{\varepsilon^3}$.
- If $\frac{2}{\varepsilon^3} < \frac{1}{2}|V(C)|$, then we obtain a longer path.
- Set ε large enough so that $\frac{2}{\varepsilon^3} < \frac{\varepsilon^2 n}{4}$ to avoid this case.

Partial Transversal Refinement



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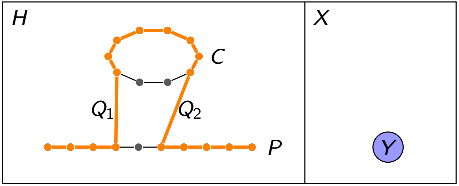
Partial Transversal Refinement



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- *If $\kappa(G) > m^2$ and R is a connected m-edge multigraph, then $\tau_R(G) \leq 8m^{5/4}n^{3/4}$.*

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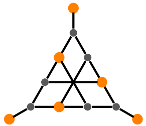

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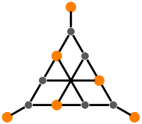

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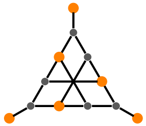

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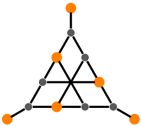

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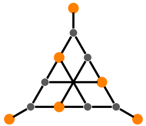

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Theorem

For each positive k, there exists n_0 such that if G is an n-vertex k-connected graph with $n \geq n_0$ and $\alpha(G) \leq k + 2$, then $\tau(G) = 1$.

Example: $P_2 + 2P_1$ is a fixer

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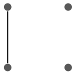

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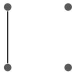

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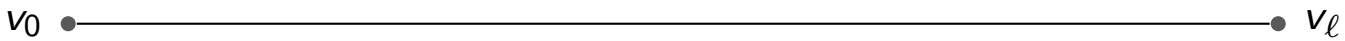

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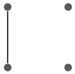

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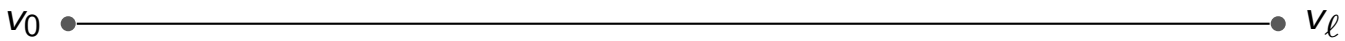

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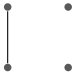

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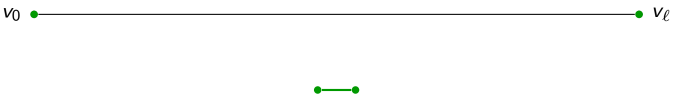

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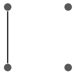

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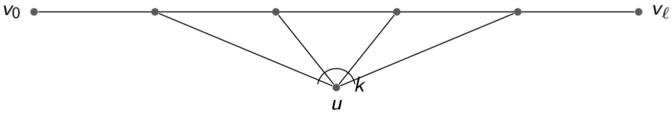

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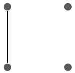

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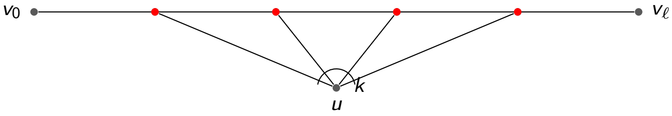

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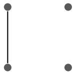

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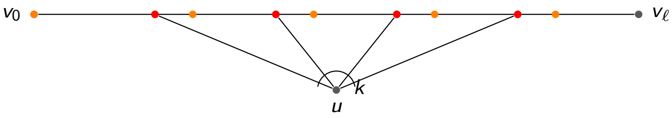

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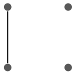

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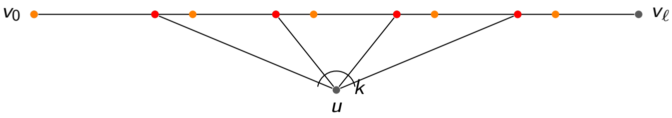

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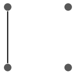

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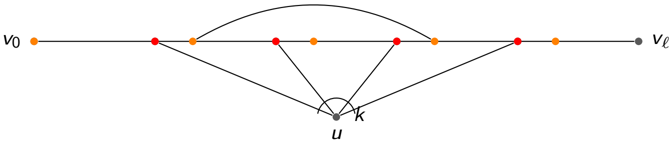

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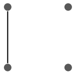

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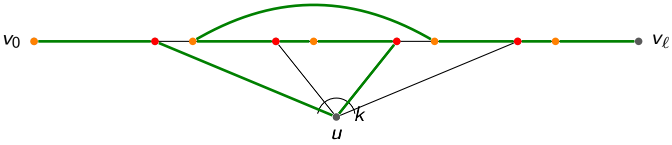

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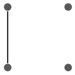

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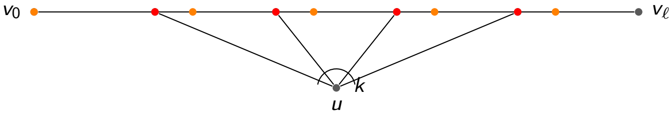

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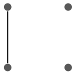

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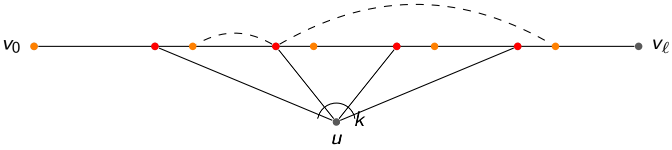

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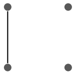

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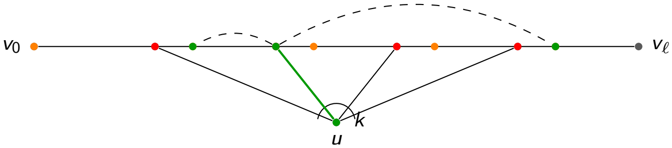

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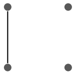

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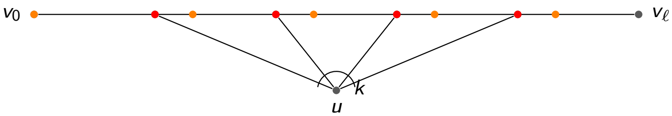

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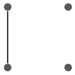

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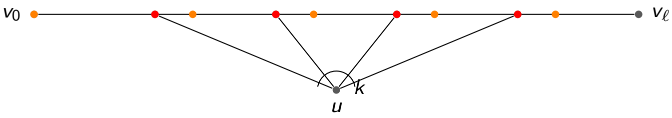

- Each vertex in S has at most 1 non-neighbor in T.
- Each vertex in S has at least k neighbors in T.
- Each vertex in S has degree at least $k+1$, and $k+1 > \Delta(G)$.

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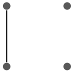

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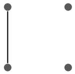

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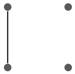

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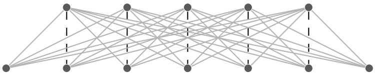

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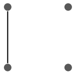

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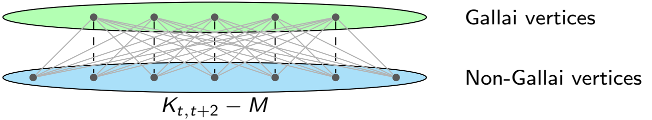

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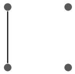

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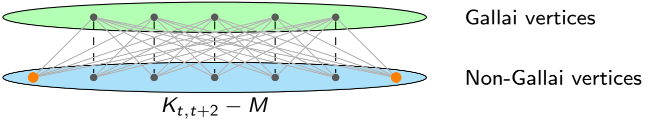

- Two non-Gallai vertices of degree $\Delta(G) - 1$.

Potential fixers

Lemma

If H is a fixer, then H is a linear forest on at most 9 vertices.

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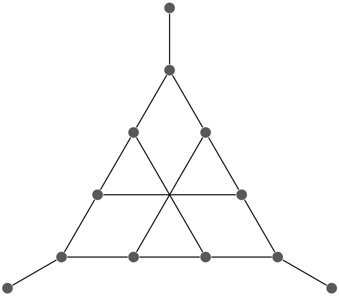

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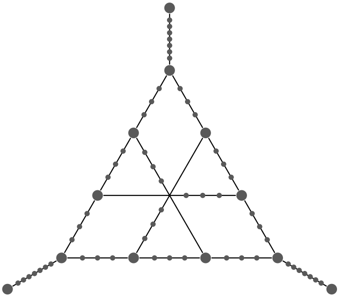

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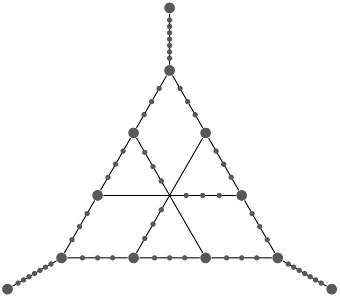

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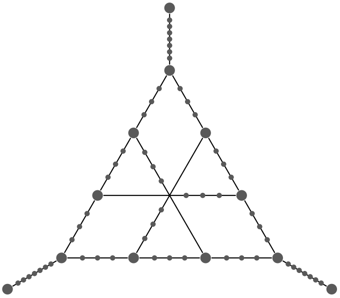

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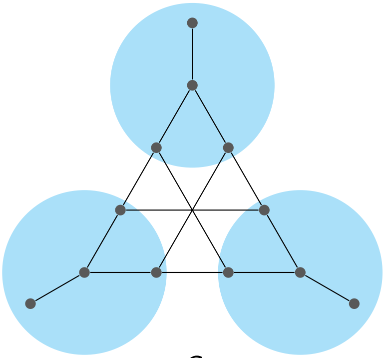

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- $|V(H)| \leq |V(G)| - 3 = 9$.

Fixers of order 4

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- All degree conditions are sharp, except that possibly $d(u) \geq \Delta(G) - 1$ is sufficient in the case of $2P_2$-free graphs.

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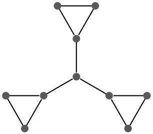

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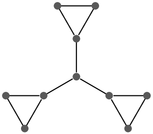

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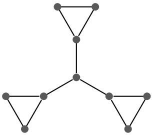

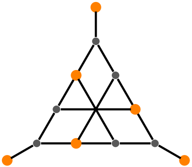

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- The 12-vertex counter-example has independence number 6.

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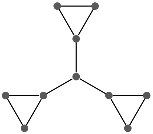

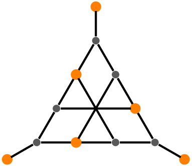

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- Is $6P_1$ a fixer?

Summary and Open Problems

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Let R be a connected m-edge multigraph. If G is an n-vertex graph and $\kappa(G) > m^2$, then $\tau_R(G) \leq 8m^{5/3}n^{3/4}$.

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Thank You.