Longest Path and Cycle Transversals in Chordal Graphs

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- $R$ and the longer parts of $P$ and $Q$ form a longer path.
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Gallai’s Problem

- If $P$ and $Q$ are longest paths in a connected graph, then $V(P) \cap V(Q) \neq \emptyset$. 

Gallai (1966): Is some vertex common to every longest path?


Counter-example (Walther–Voss; T.I. Zamfirescu)
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Examples of Gallai families:
- Klavžar–Petkovšek (1990): split graphs and cacti
- BGLS+Joos (2015): circular arc graphs
- Jobson–Kézdy–Lehel–White (2016): $P_4$-free graphs, dually chordal graphs
- Cerioli–Lima (2016): $P_4$-sparse graphs
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Question

Do the chordal graphs form a Gallai family?
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Grüenbaum (1974): Some connected graph $G$ has $\text{lpt}(G) = 3$. 

Open: is there a connected graph $G$ with $\text{lpt}(G) \geq 4$?
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Rautenbach–Sereni (2014): $lpt(G) \leq \frac{n^4 - n^2}{90}$.  
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**Theorem**

*If $G$ is a connected $n$-vertex chordal graph, then $\text{lpt}(G) \leq O(\log^2 n)$.***
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- Let $\text{lct}(G)$ be the min. size of a longest cycle transversal in $G$.
- Connected graphs can have $\text{lct}(G)$ linear in $|V(G)|$: 

![Diagram showing a tree with multiple branches and vertices representing the concept of longest cycle transversals.](image-url)
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- If $G$ is 2-connected, then the longest cycles in $G$ pairwise intersect.
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Theorem

If $G$ is a 2-connected $n$-vertex chordal graph, then $\text{lct}(G) \leq O(\log n)$. 

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*Note: The text was formatted to ensure proper indentation and spacing for a clear and readable representation.*
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Open: is there a 2-connected graph $G$ with $\text{lct}(G) \geq 4$?

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Longest Cycle Transversals

- A longest cycle transversal of $G$ is a set of vertices that intersects every longest cycle.
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Theorem

If $G$ is a 2-connected $n$-vertex chordal graph, then $lct(G) \leq O(\log n)$. 
Tree Representations

- Gavril (1974): A graph $G$ is \textit{chordal} if and only if $G$ in the intersection graph of subtrees of a \textit{host tree} $T$.  

Given $u \in V(G)$, let $S(u)$ be the corresponding subtree in $G$.

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Fact: if $T$ is a minimal tree representation of a chordal graph $G$, then the bags in $T$ are the maximal cliques in $G$ and $|V(T)| \leq \#(\text{max. cliques in } G)$.

Lemma: every tree $T$ has a vertex $z$ such that each component of $T - z$ has at most $|V(T)|/2$ vertices.

Lemma: If $G$ is 2-connected and $C_1$ and $C_2$ are longest cycles in $G$, then $C_1 \cup C_2$ is a 2-connected subgraph.
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![Diagram of tree $T$ and graph $G$]
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The Core Capture Property

Let \( G \) be chordal with tree representation \( T \). The core of a subgraph \( H \) of \( G \) is given by

\[
\text{core}(H) = \bigcup_{uv \in E(H)} (V(S(u)) \cap V(S(v))).
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A set $W \subseteq V(T)$ has the core capture property (ccp) for a family of subgraphs $H$ if each $H \in H$ has a core vertex in $W$. 
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Key Lemma

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Let:
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1. \( G \): a 2-connected chordal graph
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3. \( C \): a family of longest cycles in \( G \)

Then there exists:

1. \( C' \): a subfamily of \( C \)
2. \( X' \): a rooted subtree of \( X \) having the ccp for \( C' \), with \(|V(X')| \leq |V(X)|/2\)
3. \( A \): a set of at most 4 verts in \( G \) meeting each cycle in \( C - C' \)

**Theorem**

If \( G \) is a 2-connected \( n \)-vertex chordal graph, then \( lct(G) \leq 4(1 + \lceil \lg n \rceil) \).
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$x$

$\rightarrow$
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- $Q$: the $xz$-path.
- $D(y)$: all descendants of $y$ (including $y$).
- $Q_0$: minimal subpath of $Q$ whose descendants $D(Q_0)$ have ccp for $C$. 
Lemma: \( G \) has distinct vertices \( w_1 \) and \( w_2 \) with \( S(w_1), S(w_2) \) containing \( Q_0 \).
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We add \( w_1, w_2 \) to \( A \).
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We add $w_1, w_2$ to $A$.

Let $C_1 = \{ C \in \mathcal{C} : V(C_1) \cap \{w_1, w_2\} = \emptyset \}$. 

A path $P$ in $G$ is good if $|V(P)| \geq 3$, $S(u) \subseteq D(Q_0) - V(Q_0)$ for each interior vertex $u \in V(P)$, and $S(v) \cap V(Q) \neq \emptyset$ for each endpoint $v$ of $P$. 

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Let $P$ be a longest good path in $G$ with endpoints $w_3$ and $w_4$.

If $C \in C_1$ does not intersect $P$, then we get a longer cycle by replacing the interior of a good path in $C$ with $w_1Pw_2$. 
Let $A = \{w_1, w_2, w_3, w_4\}$ and $C' = \{C \in C : V(C) \cap A = \emptyset\}$. 
Key Lemma Sketch

Let $A = \{w_1, w_2, w_3, w_4\}$ and $C' = \{C \in C : V(C) \cap A = \emptyset\}$.

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Take \( X' \) to be this component.
Host tree: subdivided star

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- Gavril (1974): A graph $G$ is **chordal** if and only if $G$ is the intersection graph of subtrees of a host tree.
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- **Long–Milans–Wigal:** if $G$ is connected and has a tree representation $T$ such that $T$ is a subdivided star, then $lpt(G) = 1$. 

![Subdivided star graph](image)
Open Problems

1. Prove \( \text{lpt}(G) = 1 \) when \( G \) is a connected chordal graph (or find a counterexample).
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3. Improve the bounds $\text{lpt}(G) \leq O(\log n)$ and $\text{lct}(G) \leq O(\log n)$ when $G$ is an $n$-vertex connected/2-connected chordal graph.

4. Our arguments do not give efficient algorithms for finding the transversals; can we find these in polynomial time?

5. Improve the Kierstead–Ren (2023+) bound $\text{lpt}(G) \leq O(n^{2/3})$ when $G$ is a connected $n$-vertex graph.

6. Find a connected graph $G$ with $\text{lpt}(G) \geq 4$ (or show no such graph exists).

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2. Prove $lpt(G) = 1$ when $G$ is a connected chordal graph with tree representation $T$, such that $T$ belongs to some nice family of trees. (When $T$ is a subdivided caterpillar, we get a constant bound on $lpt(G)$ but not $lpt(G) = 1$.)
3. Improve the bounds $lpt(G) \leq O(\log^2 n)$ and $lct(G) \leq O(\log n)$ when $G$ is an $n$-vertex connected/2-connected chordal graph.
4. Our arguments do not give efficient algorithms for finding the transversals; can we find these in polynomial time?
5. Improve the Kierstead–Ren (2023+) bound $lpt(G) \leq O(n^{2/3})$ when $G$ is a connected $n$-vertex graph.
6. Find a connected graph $G$ with $lpt(G) \geq 4$ (or show no such graph exists).
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Thank You.