Longest Path and Cycle Transversals in Chordal Graphs

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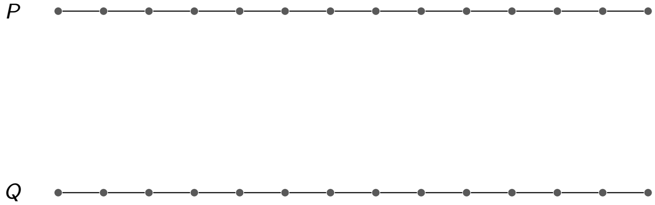

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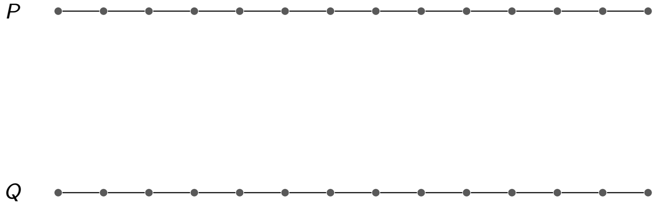

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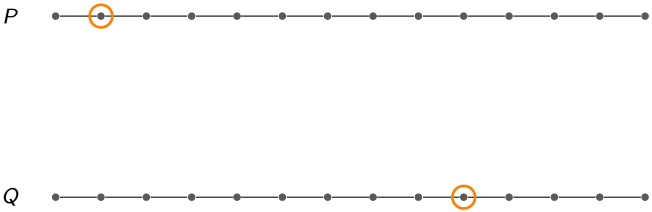

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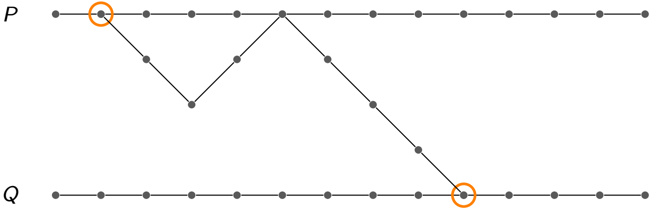

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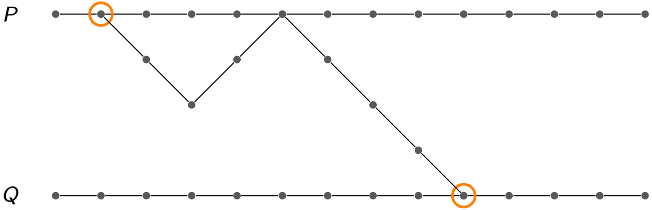

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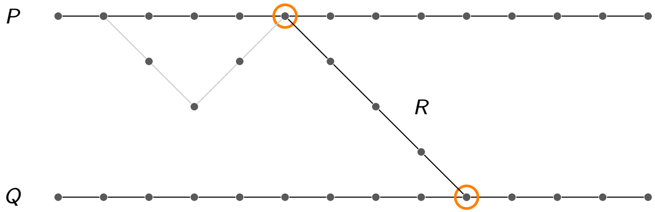

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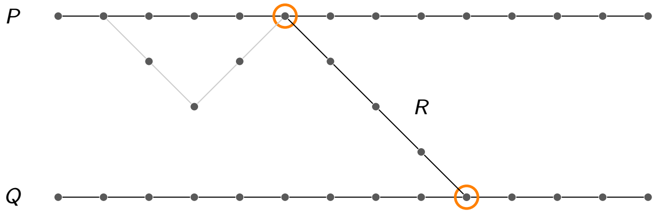

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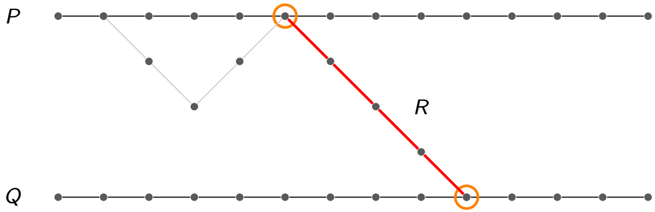

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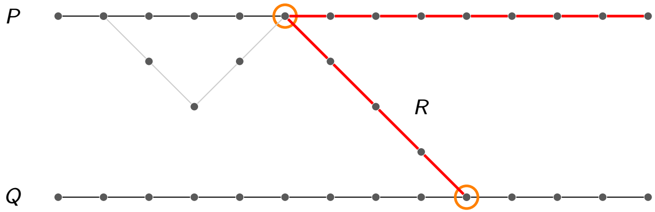

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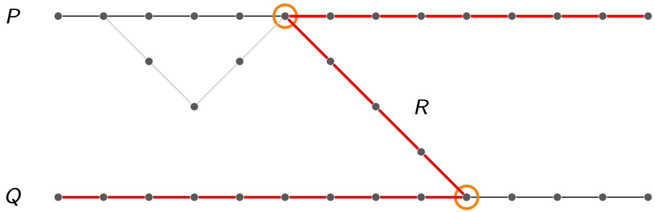

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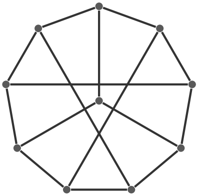

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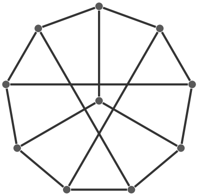

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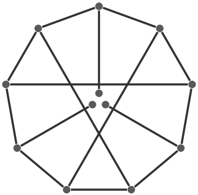

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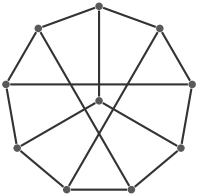

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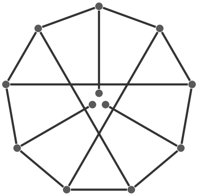

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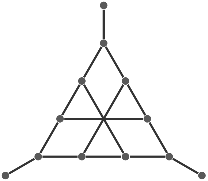

Redrawn

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Question

Do the chordal graphs form a Gallai family?

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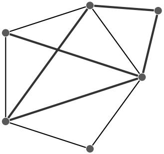
- Open: is there a connected graph G with $\mathrm{lpt}(G) \geq 4$?

Chordal Graphs



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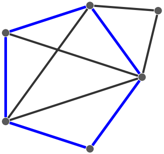

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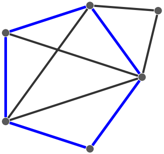

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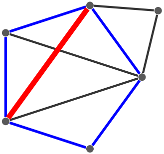

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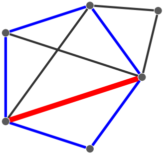

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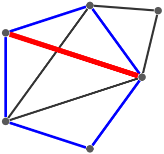

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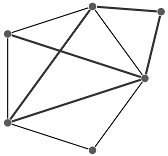

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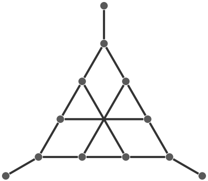

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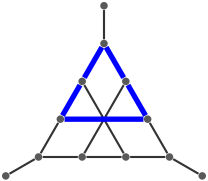

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If G is a connected n-vertex chordal graph, then $\mathrm{lpt}(G) \leq O(\log^2 n)$.

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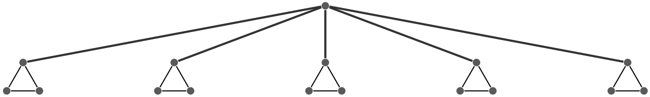

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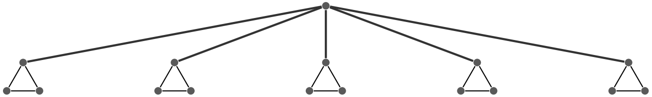

- If G is 2-connected, then the longest cycles in G pairwise intersect.

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Theorem

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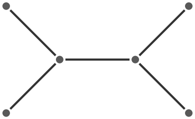

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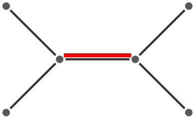

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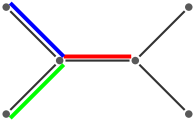

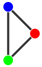

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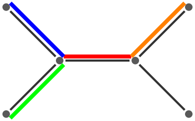

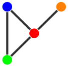

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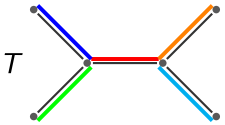

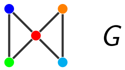

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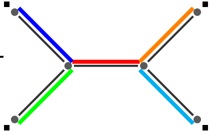
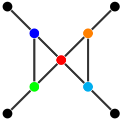
T 

 G

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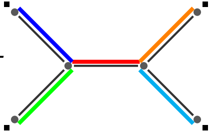
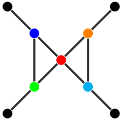
T 

 G

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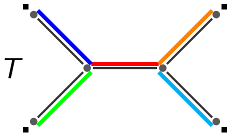

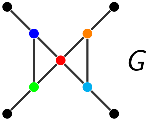

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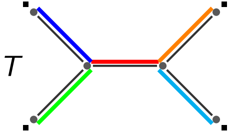

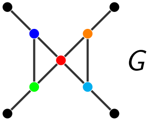

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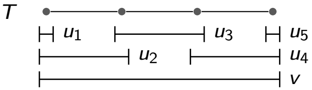

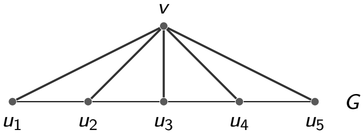

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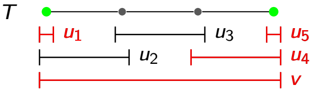

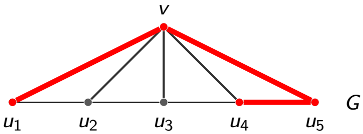

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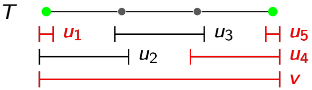

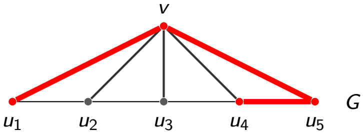

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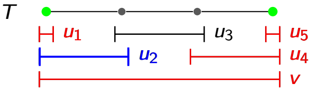

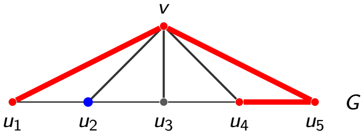

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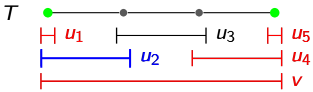

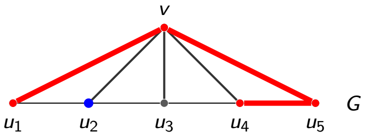

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- A set $W \subseteq V(T)$ has the core capture property (ccp) for a family of subgraphs $\mathcal{H}$ if each $H \in \mathcal{H}$ has a core vertex in W.

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If G *is a 2-connected* n*-vertex chordal graph, then* $\mathrm{lct}(G) \leq 4(1 + \lceil \lg n \rceil)$*.*

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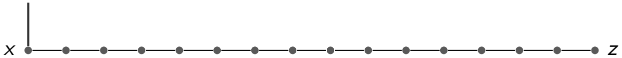

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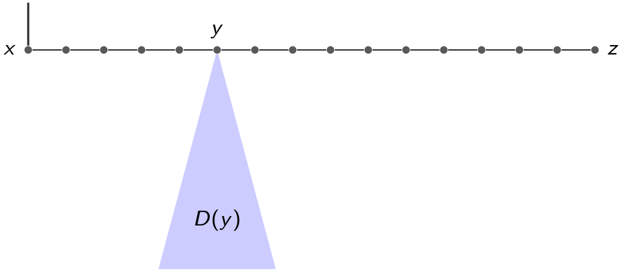

- $\mathcal{C}$: family of longest cycles.
- X: rooted subtree with root x, having ccp for $\mathcal{C}$.
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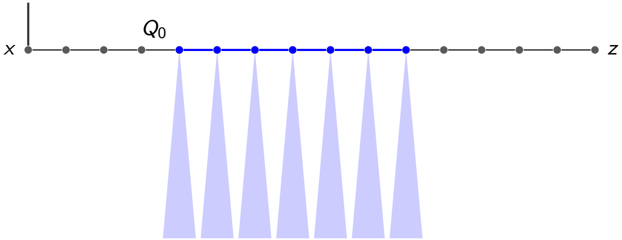

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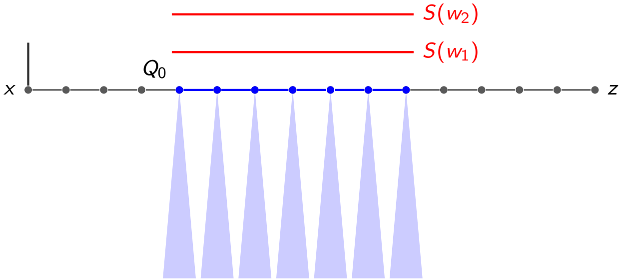

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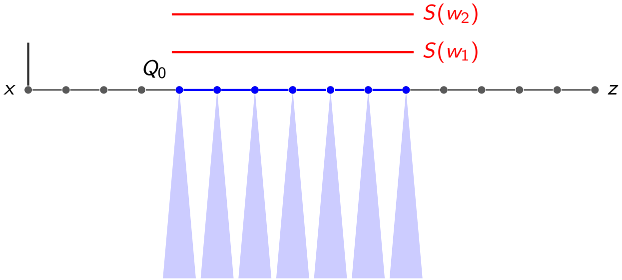

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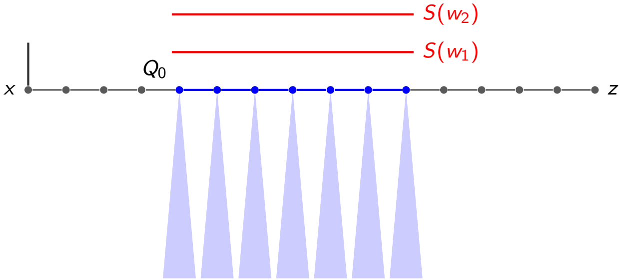

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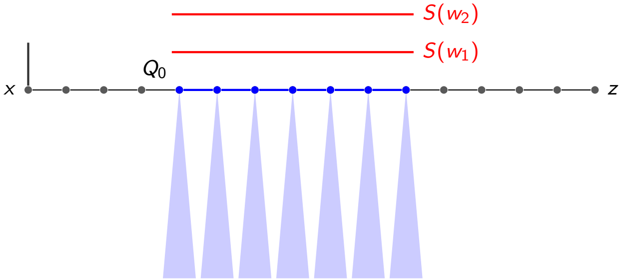

- A path P in G is good if $|V(P)| \geq 3$, $S(u) \subseteq D(Q_0) - V(Q_0)$ for each interior vertex $u \in V(P)$, and $S(v) \cap V(Q) \neq \varnothing$ for each endpoint v of P.

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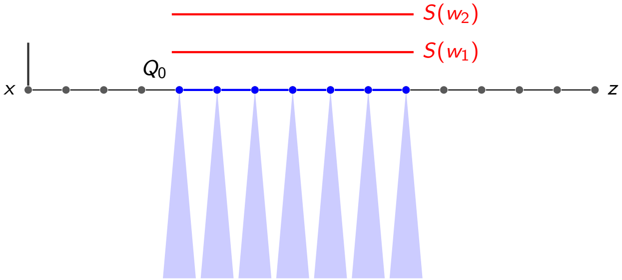

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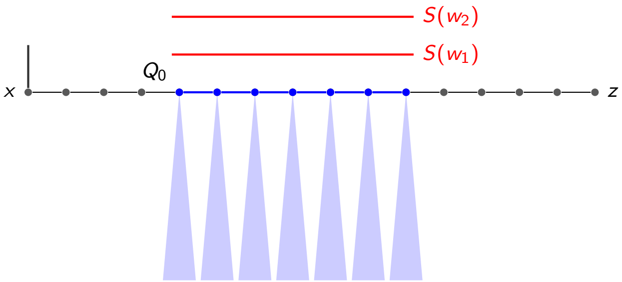

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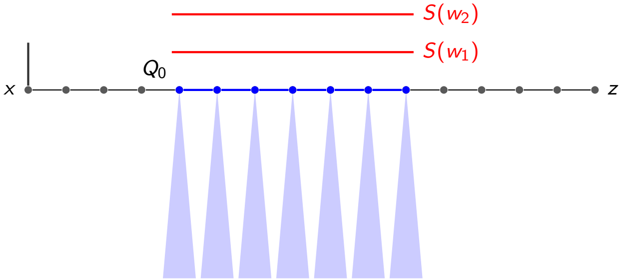

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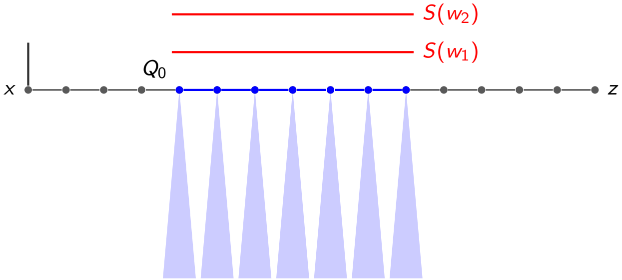

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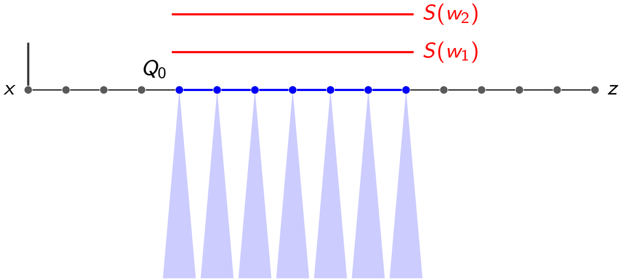

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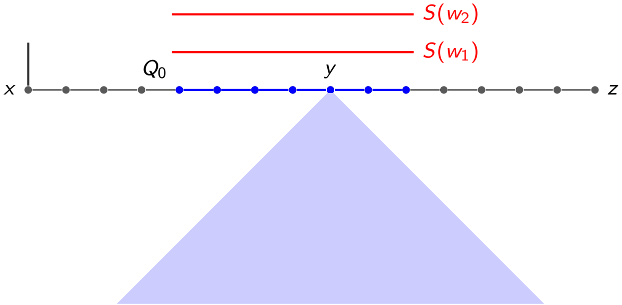

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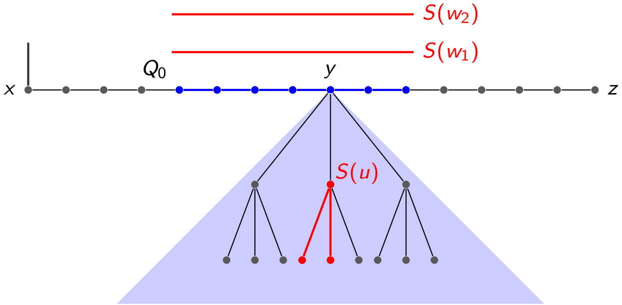

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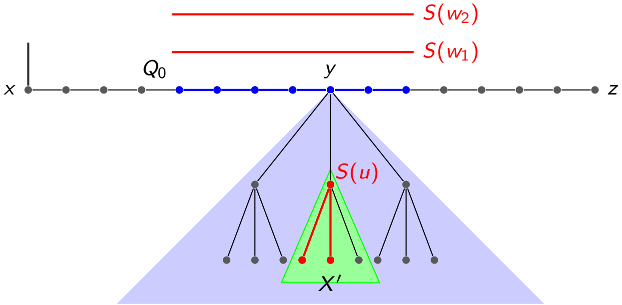

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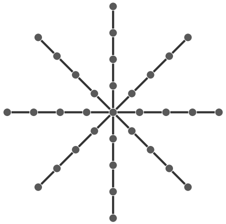

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