A Dichotomy Theorem for First-Fit Chain Partitions

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Abstract

First-Fit is a greedy algorithm for partitioning the elements of a poset into chains. Let $\text{FF}(w, Q)$ be the maximum number of chains that First-Fit uses on a $Q$-free poset of width $w$. A result due to Bosek, Krawczyk, and Matecki states that $\text{FF}(w, Q)$ is finite when $Q$ has width at most 2. We describe a family of posets $Q$ and show that the following dichotomy holds: if $Q \in \mathcal{Q}$, then $\text{FF}(w, Q) \leq 2^{c \log w^2}$ for some constant $c$ depending only on $Q$, and if $Q \notin \mathcal{Q}$, then $\text{FF}(w, Q) \geq 2^w - 1$.

1 Introduction

A partially ordered set or poset is a pair $(P, \leq)$ where $P$ is a set and $\leq$ is an antisymmetric, reflexive, and transitive relation on $P$. We use $P$ instead of $(P, \leq)$ when there is no ambiguity in simplifying this notation. We write $x > y$ when $x \geq y$ and $x \neq y$. All posets in this paper are finite.

Two points $x, y \in P$ are comparable if $x \leq y$ or $y \leq x$. Otherwise, $x$ and $y$ are said to be incomparable, denoted $x \perp y$. We say that $y$ covers $x$ if $y > x$ and there does not exist a point $z \in P$ such that $y > z > x$. A chain $C$ is a set of pairwise comparable elements, and the height of $P$ is the size of a maximum chain. An antichain $A$ is a set of pairwise incomparable elements, and the width of $P$ is the size of a maximum antichain.

A chain partition of a poset $P$ is a partition of the elements of $P$ into nonempty chains. Dilworth’s theorem states that for each poset $P$, the minimum size of a chain partition equals the width of $P$. A Dilworth partition of $P$ is a chain partition of $P$ of minimum size. A poset $Q$ is a subposet of $P$ if $Q$ can be obtained from $P$ by deleting elements. We say that $P$ is $Q$-free if $Q$ is not a subposet of $P$.

First-Fit is a simple algorithm that constructs an ordered chain partition of a poset $P$ by processing the elements of $P$ in a given presentation order. Suppose that First-Fit has already partitioned $\{x_1, \ldots, x_{k-1}\}$ into chains $(C_1, \ldots, C_t)$. First-Fit then assigns $x_k$ to the first chain $C_j$ such that $C_j \cup \{x_k\}$ is a chain; if necessary, we introduce a new chain $C_{t+1}$ containing only $x_k$.

We are concerned with the efficiency of the First-Fit algorithm. A classical example due to Kierstead (see, for example, pages 87 and 88 in [13]) shows that First-Fit may use arbitrarily
many chains even on posets of width 2. However, Bosek, Krawczyk, and Matecki [4] proved that for each fixed poset \( Q \) of width at most 2, the number of chains used by First-Fit on a \( Q \)-free poset \( P \) is bounded in terms of the width of \( P \). Let \( \text{FF}(w, Q) \) be the maximum, over all \( Q \)-free posets \( P \) of width \( w \) and all presentation orders of \( P \), of the number of chains that First-Fit uses. The upper bound on \( \text{FF}(w, Q) \) given by Bosek, Krawczyk, and Matecki’s can be as large as a tower of \( w \)'s with a height that is linear in \(|Q|\).

1.1 Prior work

Aside from the result of Bosek, Krawczyk, and Matecki [4], prior work has focused on establishing bounds on \( \text{FF}(w, Q) \) when \( Q \) is a particular poset of interest. We outline the history briefly.

Let \( N \) be the 4-element poset with points \( \{a, b, c, d\} \) and relations \( a < c \) and \( b < c, d \). The performance of First-Fit on \( N \)-free posets is closely related to the performance of the greedy coloring algorithm on graphs that contain no induced copies of the 4-vertex path. The clique number of a graph \( G \), denoted \( \omega(G) \), is the maximum size of a set of pairwise adjacent vertices in \( G \). A proper coloring of \( G \) assigns to each vertex a color such that adjacent vertices receive distinct colors. The greedy coloring algorithm gives a proper coloring of \( G \) by processing the vertices of \( G \) in some order, greedily assigning to each vertex \( u \) the first color not already assigned to a neighbor of \( u \). Extending our notation to the analogous problem for graphs, let \( \text{FFG}(w, H) \) be the maximum, over all graphs \( G \) such that \( G \) contains no induced copy of \( H \) and \( \omega(G) \leq w \) and all orderings of the vertices of \( G \), of the number of colors used by the greedy coloring algorithm. Let \( P_n \) be the path on \( n \) vertices. It is well-known that \( \text{FFG}(w, P_4) = w \). If \( P \) is a poset and \( G \) is the incomparability graph of \( P \), then \( P \) contains \( N \) as a subposet if and only if \( G \) contains an induced copy of \( P_4 \). Hence we have \( w \leq \text{FF}(w, N) \leq \text{FFG}(w, P_4) = w \) and so \( \text{FF}(w, N) = w \). Kierstead, Penrice, and Trotter [14] proved that \( \text{FFG}(w, P_5) \) is bounded by a function of \( w \), and a consequence of a theorem of Gyárfás and Lehel [8] is that \( \text{FFG}(w, P_6) \) is unbounded. As noted in [14], combining results in these two papers gives that, when \( T \) is a tree, \( \text{FFG}(w, T) \) is bounded if and only if \( T \) does not contain \( P_2 + 2P_1 \) as an induced subgraph, where \( P_2 + 2P_1 \) is the disjoint union of a copy of \( P_2 \) and two copies of \( P_1 \).

Let \( r \) denote the chain with \( r \) elements. The disjoint union of posets \( P \) and \( Q \) is denoted \( P + Q \), with each element in \( P \) incomparable to every element in \( Q \). An interval order is a poset whose elements are closed intervals with \([x_1, x_2] < [y_1, y_2]\) if and only if \( x_2 < y_1 \). Fishburn [7] proved that a poset \( P \) is an interval order if and only if \( P \) is \((2 + 2)\)-free. The problem of determining the performance of First-Fit on interval orders is still open, despite significant efforts by various different research groups over the years. Currently, the best known bounds are \((5 - o(1))w \leq \text{FF}(w, 2 + 2) \leq 8w \). The lower bound is due to Kierstead, D. Smith, and Trotter [11]. The upper bound is due to Brightwell, Kierstead, and Trotter (unpublished), and independently Narayanaswamy and Babu [16], who improved on the breakthrough column construction method due to Penmaraju, Raman, and Varadarajan [17].

The interval orders are the \((2 + 2)\)-free posets; we obtain a larger class of posets by
forbidding the disjoint union of longer chains. Bosek, Krawczyk, and Szczypka \cite{bosek_krawczyk_szczypka} showed that when \( r \geq s \), \( \text{FF}(w, r + x) \leq (3r - 2)(w - 1)w + w \). Joret and Milans \cite{joret_milans} improved the bound to \( \text{FF}(w, (r + x)) \leq 8(r - 1)(s - 1)w \). Dujmović, Joret, and Wood \cite{dujmovic_joret_wood} further improved the bound to \( \text{FF}(w, (r + x)) \leq 8(2r - 3)w \), which is best possible up to the constants.

The ladder of height \( n \), denoted \( L_n \), consists of two disjoint chains \( x_1 < \cdots < x_n \) and \( y_1 < \cdots < y_n \) with \( x_i \leq y_j \) if and only if \( i \leq j \) and no relations of the form \( y_i \leq x_j \). Kierstead and M. Smith \cite{kierstead_smith} showed that \( \text{FF}(w, L_2) = w^2 \) and \( \text{FF}(2, L_n) \leq 2n \). They also proved the general bound \( \text{FF}(w, L_n) \leq w^{((\lg(w)) + \lg(n))} \), where \( \lg(x) \) denotes the base-2 logarithm; this result plays an important role in our main theorem.

1.2 Our Results

Our aim is to say something about the behavior of \( \text{FF}(w, Q) \) in terms of the structure of \( Q \). We obtain subexponential bounds on \( \text{FF}(w, Q) \) when \( Q \) belongs to a particular family of posets \( Q \), and we also give an exponential lower bound on \( \text{FF}(w, Q) \) when \( Q \not\in Q \). From the point of view of the First-Fit algorithm, efficiency is vastly improved if a single poset in \( Q \) is forbidden. From the point of view of an adversary, forcing First-Fit to use exponentially many chains requires all posets in \( Q \) to appear.

For each \( x \in P \), we define the above set of \( x \), denoted \( A(x) \), to be \( \{ y \in P : y > x \} \); also, when \( S \) is a set of points, we define \( A(S) \) to be \( \bigcup_{x \in S} A(x) \). Similarly, the below set of \( x \), denoted \( B(x) \), is \( \{ y \in P : y < x \} \) and we extend this to sets via \( B(S) = \bigcup_{x \in S} B(x) \). We define \( A[x] = A(x) \cup \{ x \} \) and similarly for \( B[x] \). The series composition of posets \( S_1, \ldots, S_n \), denoted \( S_1 \oplus \cdots \oplus S_n \), produces a poset \( S \) which has disjoint copies of \( S_1, \ldots, S_n \) arranged so that \( x < y \) whenever \( x \in S_i, y \in S_j \) and \( i < j \). The blocks of \( S \) are the subposets \( S_1, \ldots, S_n \).

2 Dichotomy Theorem

A poset is ladder-like if its elements can be partitioned into two chains \( C_1 \) and \( C_2 \) such that if \( (x, y) \in C_1 \times C_2 \) and \( x \) is comparable to \( y \), then \( x < y \). Our first lemma shows that every ladder-like poset is contained in a sufficiently large ladder.

Lemma 1. If \( P \) is a ladder-like poset of size \( n \), then \( P \) is a subposet of \( L_n \).

Proof. Let \( P \) be a ladder-like poset of size \( n \). Clearly the 1-element poset is a subposet of \( L_1 \), and so we may assume \( n \geq 2 \). Let \( C_1 \) and \( C_2 \) be a chain partition of \( P \) such that whenever \( (x, y) \in C_1 \times C_2 \) and \( x \) and \( y \) are comparable, we have \( x < y \). Suppose that \( P \) has a maximum element \( u \). Recall that \( L_n \) consists of chains \( x_1 < \cdots < x_n \) and \( y_1 < \cdots < y_n \) with \( x_i \leq y_j \) if and only if \( i \leq j \). By induction, \( P - u \) can be embedded into the copy of \( L_{n-1} \) in \( L_n \) induced by \( \{x_1, \ldots, x_{n-1}\} \cup \{y_1, \ldots, y_{n-1}\} \). Allowing \( y_n \) to play the role of \( u \) completes a copy of \( P \) in \( L_n \). Next, suppose that \( P \) has no maximum element. Let \( u = \max C_2 \), let \( S = \{v \in C_1 : v \parallel u\} \), and let \( s = |S| \). Since \( P \) has no maximum element, it follows that \( s \geq 1 \). By induction, \( P - S \) can be embedded in the copy of \( L_{n-s} \) in \( L_n \) induced by
\{x_1, \ldots, x_{n-s}\} \cup \{y_1, \ldots, y_{n-s}\}. Allowing \{x_{n-s+1}, \ldots, x_n\} to play the role of \(S\) completes a copy of \(P\) in \(L_n\). \qed

The performance of First-Fit on a poset \(P\) can be analyzed using a static structure. A \textit{wall} of a poset \(P\) is an ordered chain partition \((C_1, \ldots, C_i)\) such that for each element \(x \in C_j\) and each \(i < j\), there exists \(y \in C_i\) such that \(y \parallel x\). It is clear that every ordered chain partition produced by First-Fit is a wall, and conversely, each wall \(W\) of \(P\) is output by First-Fit when the elements of \(P\) are presented in order according to \(W\). Hence, the worst-case performance of First-Fit on \(P\) is equal to the maximum size of a wall in \(P\). A \textit{subwall} of a wall \(W\) is obtained from \(W\) by deleting zero or more of the chains in \(W\). Note that if \(W\) is a wall of \(P\), then each subwall of \(W\) is a wall of the corresponding subposet of \(P\).

For each positive integer \(k\), we construct a poset called the \textit{reservoir} of width \(k\), denoted \(R_k\), and a corresponding wall \(W_k\) of size \(2^k - 1\). The reservoirs provide an example of a family of posets which are good at avoiding subposets and yet still have exponential First-Fit performance.

**Theorem 2.** For each \(k \geq 1\), the reservoir \(R_k\) has width \(k\) and a wall \(W_k\) of size \(2^k - 1\).

\textit{Proof.} Let \(R_1\) be the 1-element poset, and let \(W_1\) be the chain partition of \(R_1\). For \(k \geq 2\), we first construct \(R_k\) using \(R_{k-1}\) and \(W_{k-1}\). Then, we give a presentation order for \(R_k\) which forces First-Fit to use at least \(2^k - 1\) chains. Let \(W_{k-1} = (C_1, \ldots, C_m)\) where \(m = 2^{k-1} - 1\), and for \(0 \leq i \leq m\), let \(\hat{S}_i\) be the subwall \((C_1, \ldots, C_i)\) with corresponding subposet \(S_i\). (Although \(S_0\) and \(\hat{S}_0\) are empty, they are convenient for describing \(R_k\).) Let \(S\) be the series composition of disjoint copies of \(S_m, S_{m-1}, \ldots, S_0\), and \(R_{k-1}\) in this order, so that \(S = S_m \circ S_{m-1} \circ \cdots \circ S_0 \circ R_{k-1}\). The poset \(R_k\) consists of a copy of \(S\) and a chain \(X\) where \(X = \{x_{m+1} < \cdots < x_1\}\) and each \(x_i\) satisfies \(A(x_i) \cap S = \emptyset\) and \(B(x_i) \cap S = S_i \cup \cdots \cup S_m\). See Figure \[\]

Note that since \(S\) is a series composition of posets of width at most \(k - 1\), it follows that \(S\) has width at most \(k - 1\). Adding \(X\) increases the width by at most 1, and so \(R_k\) has width at most \(k\). An antichain in the top copy of \(R_{k-1}\) of size \(k - 1\) and \(x_1\) form an antichain in \(R_k\) of size \(k\).

It remains to show that First-Fit might use as many as \(2^k - 1\) chains to partition \(R_k\). Consider the partial presentation order given by \(\hat{S}_m, x_{m+1}, \hat{S}_{m-1}, x_m, \ldots, \hat{S}_1, x_2, \hat{S}_0, x_1\). We claim that First-Fit assigns color \(j\) to \(x_j\) for \(1 \leq j \leq m + 1\). Indeed, when \(\hat{S}_{j-1}\) is presented, the points in \(S_{j-1}\) are above all previously presented points except \(\{x_{j+1}, \ldots, x_{m+1}\}\), which have already been assigned colors larger than \(j\). It follows that First-Fit uses colors \(\{1, \ldots, j - 1\}\) on \(S_{j-1}\). Next, \(x_j\) is presented; since \(x_j\) is above all previously presented points except those in \(S_{j-1}\), it follows that First-Fit assigns color \(j\) to \(x_j\).

In the final stage, we present the top copy of \(R_{k-1}\) in order given by \(W_{k-1}\). This copy of \(R_{k-1}\) is incomparable to each point in \(X\) and it follows that First-Fit uses \(m\) new colors on these points. In total, First-Fit uses \(m + 1 + m\) colors, and \(2m + 1 = 2^k - 1\). \qed

If \(Q\) is a poset such that \(\text{FF}(w, Q)\) is subexponential in \(w\), then Theorem \[2]\ implies that \(Q\) is a subposet of a sufficiently large reservoir \(R_k\). These posets have a nice description.
Definition 3. Let $Q$ be the minimal poset family which contains the ladder-like posets and is closed under series composition.

Our next lemma shows that $Q$ characterizes the posets of width 2 that appear in reservoirs.

Lemma 4. Let $Q$ be a poset of width 2. Some reservoir $R_k$ contains $Q$ as a subposet if and only if $Q \in Q$.

Proof. If $Q$ is ladder-like and has $t$ elements, then $Q$ is a subposet of $L_t$ by Lemma 1 and $L_t$ is a subposet of a sufficiently large reservoir. Suppose that $Q = Q_1 \oplus Q_2$ for some $Q_1, Q_2 \in Q$ with $|Q_1|, |Q_2| < |Q|$. By induction, $Q_1$ and $Q_2$ are subposets of $R_k$ for some $k$. Since $R_{k+1}$ contains the series composition of two copies of $R_k$, it follows that $Q$ is a subposet of $R_{k+1}$.

Let $Q$ be a poset of width 2 that is contained in some reservoir. We show that $Q \in Q$ by induction on $|Q|$. Let $k$ be the least positive integer such that $Q \subseteq R_k$, and let $S_0, \ldots, S_m$, $S$, and $X$ be as in the definition of $R_k$. If $Q \cap S$ is a chain, then $(Q \cap S, Q \cap X)$ is a chain partition witnessing that $Q$ is ladder-like, and so $Q \in Q$. Let $y, z$ be a maximal incomparable pair in $Q \cap S$, meaning that if $y', z' \in Q \cap S$, $y' \geq y$, $z' \geq z$ and $(y', z') \neq (y, z)$, then $y'$ and $z'$ are comparable. We claim that if $u \in Q$ and $u$ is above one of $\{y, z\}$, then $u$ is above both $y$ and $z$. This holds for $u \in Q \cap S$ by maximality of the pair $y, z$. This holds for $u \in Q \cap X$ since $y \parallel z$ implies that $y$ and $z$ belong to the same block in $S$, and all comparison relations between $u \in X$ and elements in $S$ depend only on their block in $S$.

Since $Q$ has width 2, it follows that $Q = Q_1 \oplus Q_2$ where $Q_1 = B[y] \cup B[z]$ and $Q_2 = A(y) \cup A(z)$. Unless $Q_2$ is empty and $Q_1 = Q$, it follows by induction that $Q_1, Q_2 \in Q$ and
therefore $Q \subseteq Q$ also. Suppose that no point in $Q$ is above $y$ or $z$. Since no point in $X$ is below a point in $S$, it follows that $Q \cap X = \emptyset$, or else a point in $Q \cap X$ would complete an antichain of size 3 with \{y, z\}.

Therefore $Q \subseteq S$. Note that $Q$ is not contained in one of the blocks in $S$ by minimality of $k$ since each such block is a subposet of $R_{k-1}$. It follows that $Q = Q_1 \oplus Q_2$ for posets $Q_1$ and $Q_2$ with $|Q_1|, |Q_2| < |Q|$. By induction, $Q_1, Q_2 \subseteq Q$ and so $Q \subseteq Q$ also. □

As a consequence of Lemma 4 and Theorem 2, it follows that $\text{FF}(w, Q) \geq 2^w - 1$ when $Q \not\subseteq Q$. It turns out that the performance of First-Fit is subexponential when $Q \subseteq Q$. Our next theorem shows how upper bounds on $\text{FF}(w, Q_1)$ and $\text{FF}(w, Q_2)$ can be used to obtain an upper bound on $\text{FF}(w, Q_1 \oplus Q_2)$. A Dilworth coloring of a poset $P$ of width $w$ is a function $\varphi: P \to [w]$, where $[w] = \{1, \ldots, w\}$ such that the preimages of $\varphi$ form a Dilworth partition.

**Theorem 5.** Let $Q_1$ and $Q_2$ be posets, let $w$, $s$, and $t$ be integers such that $\text{FF}(w, Q_1) < s$ and $\text{FF}(w, Q_2) < t$, and let $Q = Q_1 \oplus Q_2$. We have $\text{FF}(w, Q) \leq stw^2 + (s + t)w$.

**Proof.** For an ordered chain partition $\mathcal{C}$ of a poset $P$, an ascending $\mathcal{C}$-chain is a chain $x_1 < \cdots < x_k$ such that the chain in $\mathcal{C}$ containing $x_i$ precedes the chain containing $x_j$ for $i < j$. Similarly, a descending $\mathcal{C}$-chain is a chain $x_1 > \cdots > x_k$ such that the chain in $\mathcal{C}$ containing $x_i$ precedes the chain containing $x_j$ for $i < j$. The $\mathcal{C}$-depth of a point $x$, denoted $d_\mathcal{C}(x)$, is the size of a maximum ascending $\mathcal{C}$-chain with bottom element $x$ and the $\mathcal{C}$-height of a point $x$, denoted $h_\mathcal{C}(x)$, is the size of a maximum descending $\mathcal{C}$-chain with top element $x$.

Let $P$ be a $Q$-free poset of width at most $w$, and let $\mathcal{C}$ be a wall of $P$. We show that $|\mathcal{C}| \leq stw^2 + (s + t)w$. We claim that for each $x \in P$, at least one of the inequalities $h_\mathcal{C}(x) \leq s$, $d_\mathcal{C}(x) \leq t$ holds. Otherwise, if $h_\mathcal{C}(x) \geq s + 1$ and $d_\mathcal{C}(x) \geq t + 1$, then we obtain a copy of $Q$ in $P$ as follows. Let $x > y_1 > y_2 > \cdots > y_k$ be a descending $\mathcal{C}$-chain and let $x < z_1 < z_2 < \cdots < z_l$ be an ascending $\mathcal{C}$-chain. Let $P_1$ be the subposet of $P$ consisting of all $u \in P$ such that for some $y_i$, the points $u$ and $y_i$ share a chain in $\mathcal{C}$ and $u \leq y_i$. Let $\mathcal{C}_1$ be the restriction of $\mathcal{C}$ to $P_1$ and observe that $\mathcal{C}_1$ is a wall of $P_1$. Indeed, suppose that $\mathcal{C}, \mathcal{C}' \subseteq \mathcal{C}_1$ where $\mathcal{C}$ precedes $\mathcal{C}'$, and let $(y_i, y_j) = (\max C, \max C')$. Let $v \in C'$ and note that $v$ and $y_j$ share a chain in $\mathcal{C}$. Let $u$ be a point in $P$ such that $u$ belongs to the same chain in $\mathcal{C}$ as $y_i$ and $u \parallel v$. Note that $u \leq y_i$, since otherwise $u > y_i > y_j \geq v$, contradicting $u \parallel v$. Therefore $u \in P_1$ and $u \in C$. Since $\mathcal{C}_1$ is a wall of $P_1$ of size $s$ and $s > \text{FF}(w, Q_1)$, it follows that $P_1$ contains a copy of $Q_1$. Similarly, we let $P_2$ be the subposet of $P$ consisting of all $u \in P$ such that for some $z_i$, the points $u$ and $z_i$ share a chain in $\mathcal{C}$ and $u \geq z_i$. Restricting $\mathcal{C}$ to $P_2$ gives a wall $\mathcal{C}_2$ of size $t$ analogously, and since $t > \text{FF}(w, Q_2)$, it follows that $P_2$ contains a copy of $Q_2$. Since every element in $P_1$ is less than $x$ and $x$ is less than every element in $P_2$, it follows that $P$ contains a copy of $Q$.

The lower part of $P$, denoted by $L$, is $\{x \in P : h_\mathcal{C}(x) \leq s\}$ and the upper part of $P$, denoted by $U$, is $P - L$. Note that $\{L, U\}$ is a partition of $P$, that $h_\mathcal{C}(x) \leq s$ for $x \in L$, and that $d_\mathcal{C}(x) \leq t$ for $x \in U$. Let $\mathcal{C}_U$ be the subwall of $\mathcal{C}$ consisting of all chains that are contained in $U$, and let $\mathcal{C}_{U,j}$ be the subwall of $\mathcal{C}_U$ consisting of the chains $C \in \mathcal{C}_U$ such that
\[ d_C(\min C) = j. \] We claim that the minimum elements of the chains in \( C_{U,j} \) form an antichain.

Suppose that \( C, C' \in C_{U,j} \) and that \( C \) precedes \( C' \). Since \( C \) precedes \( C' \), it is not possible for \( \min C > \min C' \). Therefore if \( \min C \) and \( \min C' \) are comparable, then it must be that \( \min C < \min C' \), and it would follow that \( d_C(\min C) > d_C(\min C') \). Hence \( |C_{U,j}| \leq w \) for \( 1 \leq j \leq t \) and so \( |C_U| \leq tw \). A symmetric argument shows that the sublist \( C_L \) consisting of all chains that are contained in \( L \) satisfies \( |C_L| \leq sw \).

It remains to bound the number of chains in \( C \) that contain points in both \( U \) and \( L \). Let \( C_{LU} \) be the sublist of \( C \) consisting of these chains. Note that for each \( C \in C \), we have that \( y, z \in C \) and \( y < z \) implies that \( h_C(y) \leq h_C(z) \) and \( d_C(y) \geq d_C(z) \). It follows that each point in \( C \cap L \) is less than each point in \( C \cap U \). Let \( \varphi \colon P \to [w] \) be a Dilworth coloring. For each \( C \in C_{LU} \) with \( y = \max(C \cap L) \) and \( z = \min(C \cap U) \), we assign to \( C \) the signature \( (\varphi(y), h_C(y), \varphi(z), d_C(z)) \). We claim that the signatures are distinct. Suppose that \( C, C' \in C_{LU} \) have the same signature and that \( C \) precedes \( C' \). Let \( y = \max(C \cap L) \), \( z = \min(C \cap U) \), \( y' = \max(C' \cap L) \), and \( z' = \min(C' \cap U) \). Note that \( y < z \) is a cover relation in \( C \) and \( y' < z' \) is a cover relation in \( C' \). Since \( \varphi(y') = \varphi(y') \), it follows that \( y, y' \) are comparable. Since \( h_C(y) = h_C(y') \), it must be that \( y < y' \). Since \( \varphi(z) = \varphi(z') \), it follows that \( z' \) and \( z \) are comparable. Since \( d_C(z') = d_C(z) \), it must be that \( z' < z \). We now have that \( y < z \) is a cover relation in \( C \) but \( y < y' < z' < z \) for points \( z', y' \) that appear in a chain \( C' \) that follows \( C \), contradicting that \( C \) is a wall.

Since the assigned signatures are distinct, we have that \( |C_{LU}| \leq stw^2 \). It follows that
\[
|C| \leq |C_{LU}| + |C_L| + |C_U| \leq stw^2 + sw + tw. \]

\[ \square \]

**Corollary 6.** Let \( Q = Q_1 \otimes \cdots \otimes Q_k \). If \( \text{FF}(w, Q_i) \leq 2^{c_i(\lg w)} \) for \( 1 \leq i \leq k \), then \( \text{FF}(w, Q) \leq 2^{(c + 6k)(\lg w)^2} \), where \( c = \sum_{i=1}^{k} c_i \).

\[ \text{Proof.} \] By induction on \( k \). For \( k = 1 \), the claim is clear. Suppose \( k \geq 2 \). Since \( \text{FF}(1, Q) \leq 1 \), we may assume \( w \geq 2 \). Let \( R = Q_1 \otimes \cdots \otimes Q_{k-1} \). By induction, \( \text{FF}(w, R) \leq 2^{(c' + 6(k-1))(\lg w)^2} \), where \( c' = \sum_{i=1}^{k-1} c_i \). By Theorem 5 with \( s = 1 + 2^{(c' + 6(k-1))(\lg w)^2} \) and \( t = 1 + 2^{c_k(\lg w)^2} \), we have \( \text{FF}(w, Q) \leq \text{FF}(w, Q) \leq sw^2 + (s + t)w \leq 3sw^2 < 2^2 \cdot 2^{(c' + 6(k-1))(\lg w)^2 + 1} + 2^{c_k(\lg w)^2 + 1} \cdot 2^{2\lg w} \). It follows that \( \lg[\text{FF}(w, Q)] < (c' + c_k + 6(k - 1))(\lg w)^2 + 4 + 2\lg w \leq (c + 6k)(\lg w)^2 \).

\[ \square \]

The following key result due to Kierstead and M. Smith \[12\] shows that First-Fit uses a subexponential number of chains on ladder-free posets. We follow with the characterization of posets \( Q \) for which \( \text{FF}(w, Q) \) is subexponential.

**Theorem 7** (Kierstead–M. Smith \[12\]). For some constant \( \gamma \), we have \( \text{FF}(w, L_n) \leq w^{\gamma(\lg w + \lg(n))} \).

**Theorem 8** (Dichotomy Theorem). Let \( Q \) be an \( n \)-element poset of width 2. If \( Q \notin Q \), then there exists a constant \( C \) (depending only on \( Q \)) such that \( \text{FF}(w, Q) \leq 2^{C(\lg w)^2} \); in fact, \( C = O(n) \) suffices. If \( Q \notin Q \), then \( \text{FF}(w, Q) \geq 2^w - 1 \).

**Proof.** Suppose \( Q \notin Q \). By Theorem 2 and Lemma 4 we have \( \text{FF}(w, Q) \geq 2^w - 1 \). Suppose that \( Q \in Q \). Since \( \text{FF}(1, Q) \leq 1 \), we may assume \( w \geq 2 \). Since \( Q \in Q \), it follows that \( Q = Q_1 \otimes \cdots \otimes Q_k \) for some ladder-like posets \( Q_1, \ldots, Q_k \). For \( 1 \leq i \leq k \), let \( n_i = |Q_i| \). Since \( Q_i \) is ladder-like, Theorem 6 implies that \( \text{FF}(w, Q_i) \leq 2^{c_i(\lg w)^2} \) where \( c_i = \gamma(1 + \frac{\lg(n_i)}{\lg w}) \leq \gamma(1 + \ldots) \).
By Corollary 6, it follows that $\text{FF}(w, Q) \leq 2^{(c+6k)(\lg w)^2}$, where $c = \sum_{i=1}^{k} c_i$. Hence, it suffices to take $C = 6k + c = 6k + \sum_{i=1}^{k} c_i \leq (6 + \gamma)k + \gamma \sum_{i=1}^{k} \lg n_i$. Since $\sum_{i=1}^{k} n_i = n$, it follows by convexity that $\sum_{i=1}^{k} \lg n_i \leq k\lg(n/k) \leq (n/e)\lg e$, where $e$ is the base of the natural logarithm. Using $k \leq n$, we conclude $C \leq (6 + \gamma)n + \gamma(n/e)\lg e = O(n)$.  

Theorem 8 provides a large separation in the behavior of First-Fit on $Q$-free posets according to whether or not $Q \in Q$. It may be that even stronger results are possible. Theorem 5 shows that if $\text{FF}(w, Q_1)$ and $\text{FF}(w, Q_2)$ are polynomial in $w$, then so is $\text{FF}(w, Q_1 \oplus Q_2)$. For large $n$, the best known lower bound on $\text{FF}(w, L_n)$ is $w^{\lg(n-1)/(n-1)}$, due to Bosek, Kierstead, Krawczyk, Matecki, and M. Smith [3]. This leaves open the possibility that $\text{FF}(w, L_n)$ is polynomial in $w$ for each fixed $n$. If so, then the separation provided by the Dichotomy Theorem would improve, yielding that $\text{FF}(w, Q)$ is polynomial when $Q \in Q$ and exponential when $Q \notin Q$.

**Question 9.** Is it true for each fixed $n$ that $\text{FF}(w, L_n)$ is bounded by a polynomial in $w$?

It is clear that $\text{FF}(w, L_1) = w$ and Kierstead and M. Smith [12] proved that $\text{FF}(w, L_2) = w^2$. Note that $L_3 = Q_1 \oplus Q_2 \oplus Q_3$ where $Q_1$ and $Q_3$ are 1-element posets and $Q_2$ is the $N$ poset. Since $\text{FF}(w, Q_1) = \text{FF}(w, Q_3) = 0$ and $\text{FF}(w, Q_2) = w$, it follows from Theorem 5 that $\text{FF}(w, L_3)$ is polynomial in $w$. A more careful analysis, along the lines of Kierstead and M. Smith’s proof of $\text{FF}(w, L_2) = w^2$, shows that $\text{FF}(w, L_3) \leq w^2(w + 1)$. Question 9 is open for $n \geq 4$.

It would also be interesting to better understand the behavior of First-Fit on $Q$-free posets when $Q \notin Q$. The smallest poset of width 2 that is not in $Q$ is the skewed butterfly, denoted $B$, which consists of the chains $x_1 < x_2 < x_3$ and $y_1 < y_2$ with relations $x_1 < y_2$ and $y_1 < x_3$. What is $\text{FF}(w, B)$?

### 3 First-Fit on Butterfly-Free Posets

The butterfly poset, denoted $B$, is $Q \oplus Q$, where $Q$ is the 2-element antichain. In this section, we obtain the asymptotics of $\text{FF}(w, B)$. The performance of First-Fit on butterfly-free posets is strongly related to the bipartite Turán number for $C_4$. Kövari, Sós, Turán [15] showed that the maximum number of edges in a subgraph of $K_{n,n}$ that excludes $C_4$ is $(1 + o(1))n^{3/2}$.

**Lemma 10** (Kövari–Sós–Turán [15]). Let $q$ be a prime power, and let $n = q^2 + q + 1$. There exists a $(q+1)$-regular spanning subgraph of $K_{n,n}$ that has no 4-cycle.

We also need a standard result about the density of primes.

**Theorem 11** (Hoheisel [9]). There exists a real number $\theta$ with $\theta < 1$ such that for all sufficiently large real numbers $x$, there is a prime in the interval $[x - x^\theta, x]$.

Since the result of Hoheisel [9], many research groups have improved the bound on $\theta$; see Baker and Harman [1] for the history. The current best bound is $\theta = 0.525$, due to Baker, Harman, and Pintz [2].
Theorem 12. \( \text{FF}(w, B) \geq (1 - o(1))w^{3/2} \).

Proof. By Theorem 11 and standard asymptotic arguments, we may assume that \( w \) has the form \( q^2 + q + 1 \), where \( q \) is prime. By Lemma 10, there exists a \( (q+1) \)-regular \((X,Y)\)-bigraph \( G \) with parts of size \( w \) that has no 4-cycle. Since \( G \) is a regular bipartite graph, it follows from Hall’s Theorem that \( G \) has a perfect matching \( M \). Let \( G' = G - M \), and let \( L \) be an ordering of \( E(G') \).

Using \( G' \), we construct a \( B \)-free poset \( P \) of width \( w \) and a wall of \( P \) size \( |E(G)| \). It will then follow that \( \text{FF}(w, B) \geq |E(G)| = (q+1)w = (1-o(1))w^{3/2} \). Let \( I_X \) be the set of all pairs \((x,e)\) such that \( x \in X, e \in E(G') \), and \( e \) is incident to \( x \). Similarly, let \( I_Y \) be the set of all pairs \((y,e)\) such that \( y \in Y, e \in E(G') \) and \( e \) is incident to \( y \). We construct \( P \) so that \( M \) is a maximum antichain, \( B(M) = I_X \), and \( A(M) = I_Y \). The subposet induced by \( I_X \cup M \) consists of \( w \) incomparable chains, indexed by \( M \). For \( x_iy_i \in M \) with \( x_i \in X \) and \( y_i \in Y \), the chain associated with \( x_iy_i \) consists of all pairs \((x_i,e)\) such that \( y_i \in E(G') \) and \( e \) is incident to \( y_i \) in reverse order according to \( L \). Note that if \( e \) is the first edge in \( L \) and \( e = xy \), then \((x,e)\) is minimal in \( P \) and \((y,e)\) is maximal. The chains in \( I_X \cup M \) and the chains in \( M \cup I_Y \) combine to form a Dilworth partition of \( P \) of size \( w \); let \( D_i \) be the Dilworth chain containing \( x_iy_i \). It remains to describe the relations between points in \( I_X \) and points in \( I_Y \). For \((x,e_1) \in I_X \) and \((y,e_2) \in I_Y \), we have that \((x,e_1)\) is covered by \((y,e_2)\) if and only if \( e_1 = e_2 = xy \in E(G') \).

We claim that \( P \) is \( B \)-free. For each element \( z \in I_X \cup M \), we have that \( B(z) \) is a chain.

Hence, a maximal element in a copy of \( B \) must belong to \( I_Y \). Similarly, since \( A(z) \) is a chain when \( z \in M \cup I_Y \), a minimal element in a copy of \( B \) must belong to \( I_X \). In a chain of cover relations from \((x,e_1) \in I_X \) up to \((y,e_2) \in I_Y \), either both points stay in the same Dilworth chain \( D_i \), implying that \( xy = x_iy_i \in M \), or there is a cover relation from a point in \( D_i \) to a point in \( D_j \), that implies \( xy = x_iy_j \) with \( x_iy_j \in E(G') \).

In both cases, \((x,e_1) \leq (y,e_2)\) implies that \( xy \in E(G) \), and it follows that a copy of \( B \) in \( P \) corresponds to a 4-cycle in \( G \), a contradiction.

It remains to construct a wall \( W \) of \( P \) of size \( |E(G)| \). The wall contains \( |E(G')| \) chains of size 2 arranged in order according to \( L \), followed by \( w \) singleton chains. For \( e \in L \) with \( e = xy \), the corresponding chain in the wall is \((x,e) < (y,e)\). These chains are followed by \( w \) singleton chains, each consisting of a point in \( M \). Let \( C_i \) and \( C_j \) be chains in \( W \) with \( i < j \), and let \( z \in C_j \). We show that \( z \) is incomparable to some point in \( C_i \). Since \( M \) is an antichain, we may assume that \( C_i \) is a chain of the form \((x,e) < (y,e)\). If \( C_j \) is a singleton chain containing only \( z \), then \( z \) is incomparable to every element in \( P \) outside its Dilworth chain. Since \((x,e)\) and \((y,e)\) are in distinct Dilworth chains, it follows that \( C_i \) contains a point incomparable to \( z \). Otherwise, \( C_j \) has the form \((x',e') < (y',e')\), and since \( i < j \), it follows that \( e \) precedes \( e' \) in \( L \). Suppose that \( z = (x',e') \). If \((x',e') \parallel (x,e)\), then \((x,e)\) is the desired point in \( C_i \). Otherwise, \((x',e')\) is comparable to \((x,e)\), implying that \((x,e)\) and \((x',e')\) are in the same Dilworth chain and \( x = x' \). Since \( e \) precedes \( e' \) in \( L \), we have \((x,e) < (x',e')\). If \((x',e')\) is also comparable to \((y,e)\), it must be that \((x',e') < (y,e)\).
But now \((x,e) < (x',e') < (y,e)\) contradicts that \((y,e)\) covers \((x,e)\) in \(P\). The case that \(z = (y',e')\) is analogous.

In a poset \(P\) with a set of elements \(S\), an extremal point of \(S\) is a minimal or maximal element in \(S\).

**Lemma 13.** Let \(C\) and \(D\) be chains in \(P\). If \(\min C \parallel \max D\) and \(\max C \parallel \min D\), then \(C\) and \(D\) are pairwise incomparable. Consequently if \(C'\) and \(D'\) are chains and \((x_1, y_1), (x_2, y_2) \in C' \times D'\) are incomparable pairs, then \(\min \{x_1, x_2\} \parallel \min \{y_1, y_2\}\) and \(\max \{x_1, x_2\} \parallel \max \{y_1, y_2\}\).

**Proof.** If \(u \leq v, u \in C\), and \(v \in D\), then \(\min C \leq u \leq v \leq \max D\). If \(u \leq v, u \in D\), and \(v \in C\), then \(\min D \leq u \leq v \leq \max C\). For the second part, either the statement is trivial or we apply the first part to the subchains of \(C'\) and \(D'\) with extremal points \(\{x_1, x_2\}\) and \(\{y_1, y_2\}\) respectively.

Starting with an arbitrary chain partition \(\mathcal{C}\), iteratively moving elements to earlier chains produces a wall \(W\) with \(|W| \leq |\mathcal{C}|\). Beginning with a Dilworth partition, it follows that each poset \(P\) of width \(w\) has a Dilworth wall consisting of \(w\) chains. If \(R\) and \(S\) are sets of points in \(P\), we write \(R < S\) if \(u < v\) when \((u,v) \in R \times S\).

**Theorem 14.** \(\text{FF}(w, B) \leq (1 + o(1))w^{3/2}\).

**Proof.** Let \(P\) be a \(B\)-free poset and let \(\mathcal{D}\) be a Dilworth wall of \(P\) with \(\mathcal{D} = (D_1, \ldots, D_w)\). Let \(R\) be the set of points \(x \in P\) such that \(A(x)\) is a chain. Let \(R' = P - R\), and note that \(B(x)\) is a chain for each \(x \in R'\) since \(P\) is \(B\)-free.

Let \(\mathcal{C}\) be a wall of \(P\) with \(\mathcal{C} = (C_1, \ldots, C_t)\); we bound \(|\mathcal{C}|\). Since \(|D| = w\), at most \(2w\) chains in \(\mathcal{C}\) contain an extremal point from a chain in \(\mathcal{D}\). Also, no two chains in \(\mathcal{C}\) are contained in the same chain in \(\mathcal{D}\), and so at most \(w\) chains in \(\mathcal{C}\) are contained in a chain in \(\mathcal{D}\). Let \(\mathcal{C}'\) be the subwall of \(\mathcal{C}\) consisting of all chains \(C \in \mathcal{C}\) that do not contain an extremal point of a chain in \(\mathcal{D}\) but contain points from at least two chains in \(\mathcal{D}\). We have that \(|\mathcal{C}| \leq |\mathcal{C}'| + 3w\). We claim that for each chain \(C_i \in \mathcal{C}'\), we have that \(C_i \cap R\) is contained in a chain in \(\mathcal{D}\). Suppose that \(C_i \cap R\) contains elements from at least two chains in \(\mathcal{D}\). Let \(D_\alpha\) be the Dilworth chain containing \(C_i\), let \(x = \max(C_i - D_\alpha)\), and let \(D_\beta\) be the Dilworth chain containing \(x\). Let \(m = \max D_\beta\), and note that \(C_i \in \mathcal{C}'\) implies \(m \notin C_i\). It follows that \(m \in C_j\) for some \(C_j \in \mathcal{C}\) with \(j \neq i\); since \(A(x)\) is a chain and \(m > x\), it follows that \(m\) is comparable to every element in \(C_i\) and therefore \(j < i\). Let \(y\) be the element covering \(x\) in \(C_i\). Note that \(y \in D_\alpha\) and \(y\) is comparable to everything in \(D_\beta\) since \(A(x)\) is a chain, and this implies \(\alpha < \beta\). Since \(m, y \in A(x)\) and \(A(x)\) is a chain, either \(m < y\) or \(m > y\). If \(m > y\), then \(m\) is comparable to everything in \(D_\alpha\), contradicting \(m \in D_\beta\) and \(\alpha < \beta\). Similarly, if \(m < y\), then \(y\) is comparable to every element in \(C_j\), contradicting \(y \in C_i\) and \(j < i\). Therefore \(C_i \cap R\) is contained in a single chain in \(\mathcal{D}\). By a symmetric argument, \(C_i \cap R'\) is contained in a single chain in \(\mathcal{D}\).

It remains to bound \(|\mathcal{C}'|\). Note that for each \(C \in \mathcal{C}'\), we have that \(C \cap R\) is contained in some Dilworth chain \(D_\alpha \in \mathcal{D}\) and \(C \cap R'\) is contained in some Dilworth chain \(D_\gamma \in \mathcal{D}\) with \(\alpha \neq \gamma\); we say that \((\alpha, \gamma)\) is the signature of \(C \in \mathcal{C}'\) if \(C \cap R \subseteq D_\alpha\) and \(C \cap R' \subseteq D_\gamma\). Note
that if $C_i, C_j \in \mathcal{C}'$ with $i < j$, then it is not possible for both $C_i$ and $C_j$ to have the same signature $(\alpha, \gamma)$, or else $C_i \cap R' < C_j < C_i \cap R$. Since the signatures are distinct, it follows that $|\mathcal{C}'| \leq w^2$ and so $\text{FF}(w, B) \leq (1 + o(1))w^2$.

Let $X$ and $Y$ be disjoint copies of $\mathcal{D}$, and let $G$ be the $(X, Y)$-bigraph in which $D_\alpha \in X$ and $D_\gamma \in Y$ are adjacent if and only if some chain in $\mathcal{C}'$ has signature $(\alpha, \gamma)$. We claim that $G$ has no 4-cycle, implying $|\mathcal{C}'| = |E(G)| \leq (1 + o(1))w^{3/2}$.

Suppose for a contradiction that $G$ has a 4-cycle on $D_\alpha, D_\beta \in X$ and $D_\gamma, D_\delta \in Y$. Let $C_i, C_j, C_k, C_\ell$ be chains in $\mathcal{C}'$ with signatures $(\alpha, \gamma), (\alpha, \delta), (\beta, \gamma)$, and $(\beta, \delta)$, respectively. Assume, without loss of generality, that $C_i$ precedes $C_j$ in $\mathcal{C}$, and let $y_1 \in C_j \cap R' \subseteq D_\delta$. Since $y_1$ is in a later chain, it must be that $x_1 \parallel y_1$ for some $x_1 \in C_i$. Since $C_j \cap R$ and $C_i \cap R$ are both contained in $D_\alpha$ and $y_1 \in C_j \cap R' < C_j \cap R < C_i \cap R$, it follows that $x_1 \in C_i \cap R' \subseteq D_\gamma$. Therefore there is an incomparable pair $(x_1, y_1) \in (C_i \cap R') \times (C_j \cap R')$. A similar argument applied to $C_k$ and $C_\ell$ with top parts in $D_\beta$ shows that there is an incomparable pair $(x_2, y_2) \in (C_k \cap R') \times (C_\ell \cap R')$. Since $C_i \cap R', C_k \cap R' \subseteq D_\gamma$ and $C_j \cap R', C_\ell \cap R' \subseteq D_\delta$, it follows from Lemma 13 that there is an incomparable pair $(x, y) \in D_\gamma \times D_\delta$ with $x \leq \min\{\max C_i \cap R', \max C_k \cap R'\}$ and $y \leq \min\{\max C_j \cap R', \max C_\ell \cap R'\}$. Similarly, there is an incomparable pair $(x', y') \in D_\alpha \times D_\beta$ with $x' \geq \max\{\min C_i \cap R, \min C_j \cap R\}$ and $y' \geq \max\{\min C_k \cap R, \min C_\ell \cap R\}$. Since $x, y < x', y'$, it follows that $\{x, y, x', y'\}$ induces a copy of $B$ in $P$.

Since $|\mathcal{C}| \leq |\mathcal{C}'| + 3w \leq (1 + o(1))w^{3/2}$, the bound on $\text{FF}(w, B)$ follows. □

**Corollary 15.** $\text{FF}(w, B) = (1 + o(1))w^{3/2}$.

The *stacked butterfly* of height $t$, denoted $B_t$, is $Q_1 \oplus \cdots \oplus Q_t$, where each $Q_i$ is a 2-element antichain. Note that $B_{2k}$ is the series composition of $k$ copies of $B$. A consequence of our results is that $\text{FF}(w, B_t)$ is bounded by a polynomial in $w$ for each fixed $t$.

**Corollary 16.** $\text{FF}(w, B_{2k}) \leq (1 + o(1))w^{3.5k-2}$

**Proof.** From Theorem 5 and Corollary 15 we have that

$$\text{FF}(w, B_{2k}) \leq (1 + o(1))w^2\text{FF}(w, B_{2(k-1)})\text{FF}(w, B_2) = (1 + o(1))w^{3.5k-2}. $$

It would be interesting to find lower bounds on $\text{FF}(w, B_{2k})$. In particular, is $\text{FF}(w, B_{2k})$ bounded below by a polynomial in $w$ whose degree grows linearly in $k$?

### 4 Conclusions and Open Problems

A consequence of Theorem 8 is that $Q$ is the family of posets $Q$ such that $\text{FF}(w, Q)$ is subexponential in $w$. It may be that $Q$ is also the family of posets $Q$ such that $\text{FF}(w, Q)$ is polynomial in $w$. This is the case if and only if Question 9 has a positive answer. Alternatively, if Question 9 has a negative answer, then it would be interesting to understand what structural properties of $Q$ lead to polynomial behavior of $\text{FF}(w, Q)$.
Problem 17. Characterize the posets $Q$ for which $\text{FF}(w,Q)$ is bounded above by a polynomial in $w$.

We have focused on upper bounds for posets in $Q$ and lower bounds for posets outside $Q$. It would be nice to obtain better bounds for posets outside $Q$. The smallest poset of width 2 that is outside $Q$ is the skewed butterfly $\hat{B}$ consisting of disjoint chains $x_1 < x_2 < x_3$ and $y_1 < y_2$ with the cover relations $x_1 < y_2$ and $y_1 < x_3$. According to Theorem 2, we have $\text{FF}(w, \hat{B}) \geq 2^w - 1$. What is $\text{FF}(w, \hat{B})$? Although Bosek, Krawczyk, and Matecki [4] provide tower-type upper bounds on $\text{FF}(w, Q)$, there may be room for significant improvement.

Question 18. Is there any poset $Q$ of width 2 for which $\text{FF}(w,Q)$ is superexponential?

We have studied the behavior of First-Fit on families that forbid a single poset $Q$, but it is also natural to ask about families that forbid a set of posets. If $\mathcal{S}$ is a set of posets, we say that a poset $P$ is $\mathcal{S}$-free if no poset in $\mathcal{S}$ is a subposet of $P$. Let $\text{FF}(w, \mathcal{S})$ be the maximum number of chains that First-Fit uses on an $\mathcal{S}$-free poset of width $w$.

Problem 19. Characterize the sets $\mathcal{S}$ for which $\text{FF}(w, \mathcal{S})$ is bounded by a polynomial in $w$.

If $\mathcal{P}$ is a poset family that is closed under taking subposets, then $\mathcal{P}$ is exactly the set of posets that is $\mathcal{S}$-free, where $\mathcal{S}$ is the set of minimal posets not in $\mathcal{P}$. A solution to Problem 19 is therefore equivalent to a characterization of all subposet-closed families $\mathcal{P}$ such that First-Fit has polynomial behavior when restricted to $\mathcal{P}$. We suspect that this is a challenging problem, but the restriction of Problem 19 to $|\mathcal{S}| \leq 2$ is likely more accessible and even partial progress would still be interesting.

References


