A Dichotomy Theorem for First-Fit Chain Partitions

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December 16, 2019

Abstract

First-Fit is a greedy algorithm for partitioning the elements of a poset into chains. Let $\mathrm{FF}(w, Q)$ be the maximum number of chains that First-Fit uses on a Q-free poset of width w. A result due to Bosek, Krawczyk, and Matecki states that $\mathrm{FF}(w, Q)$ is finite when Q has width at most 2. We describe a family of posets $\mathcal{Q}$ and show that the following dichotomy holds: if $Q \in \mathcal{Q}$, then $\mathrm{FF}(w, Q) \le 2^{c(\log w)^2}$ for some constant c depending only on Q, and if $Q \notin \mathcal{Q}$, then $\mathrm{FF}(w, Q) \ge 2^w - 1$.

1 Introduction

A *partially ordered set* or *poset* is a pair $(P, \le)$ where P is a set and $\le$ is an antisymmetric, reflexive, and transitive relation on P. We use P instead of $(P, \le)$ when there is no ambiguity in simplifying this notation. We write $x > y$ when $x \ge y$ and $x \ne y$. All posets in this paper are finite.

Two points $x, y \in P$ are *comparable* if $x \le y$ or $y \le x$. Otherwise, x and y are said to be *incomparable*, denoted $x \parallel y$. We say that y covers x if $y > x$ and there does not exist a point $z \in P$ such that $y > z > x$. A *chain* C is a set of pairwise comparable elements, and the *height* of P is the size of a maximum chain. An *antichain* A is a set of pairwise incomparable elements, and the *width* of P is the size of a maximum antichain.

A *chain partition* of a poset P is a partition of the elements of P into nonempty chains. Dilworth's theorem states that for each poset P, the minimum size of a chain partition equals the width of P. A *Dilworth partition* of P is a chain partition of P of minimum size. A poset Q is a *subposet* of P if Q can be obtained from P by deleting elements. We say that P is *Q-free* if Q is not a subposet of P.

First-Fit is a simple algorithm that constructs an ordered chain partition of a poset P by processing the elements of P in a given *presentation order*. Suppose that First-Fit has already partitioned $\{x_1, \ldots, x_{k-1}\}$ into chains $(C_1, \ldots, C_t)$. First-Fit then assigns x_k to the first chain C_j such that $C_j \cup \{x_k\}$ is a chain; if necessary, we introduce a new chain C_{t+1} containing only x_k.

We are concerned with the efficiency of the First-Fit algorithm. A classical example due to Kierstead (see, for example, pages 87 and 88 in [13]) shows that First-Fit may use arbitrarily many chains even on posets of width 2. However, Bosek, Krawczyk, and Matecki [4] proved that for each fixed poset Q of width at most 2, the number of chains used by First-Fit on a Q-free poset P is bounded in terms of the width of P. Let $\mathrm{FF}(w,Q)$ be the maximum, over all Q-free posets P of width w and all presentation orders of P, of the number of chains that First-Fit uses. The upper bound on $\mathrm{FF}(w,Q)$ given by Bosek, Krawczyk, and Matecki's can be as large as a tower of w's with a height that is linear in $|Q|$.

1.1 Prior work

Aside from the result of Bosek, Krawczyk, and Matecki [4], prior work has focused on establishing bounds on $\mathrm{FF}(w,Q)$ when Q is a particular poset of interest. We outline the history briefly.

Let N be the 4-element poset with points $\{a,b,c,d\}$ and relations $a < c$ and $b < c, d$. The performance of First-Fit on N-free posets is closely related to the performance of the greedy coloring algorithm on graphs that contain no induced copies of the 4-vertex path. The *clique number* of a graph G, denoted $\omega(G)$, is the maximum size of a set of pairwise adjacent vertices in G. A *proper coloring* of G assigns to each vertex a color such that adjacent vertices receive distinct colors. The *greedy coloring algorithm* gives a proper coloring of G by processing the vertices of G in some order, greedily assigning to each vertex u the first color not already assigned to a neighbor of u. Extending our notation to the analogous problem for graphs, let $\mathrm{FFG}(w,H)$ be the maximum, over all graphs G such that G contains no induced copy of H and $\omega(G) \leq w$ and all orderings of the vertices of G, of the number of colors used by the greedy coloring algorithm. Let P_n be the path on n vertices. It is well-known that $\mathrm{FFG}(w,P_4) = w$. If P is a poset and G is the incomparibility graph of P, then P contains N as a subposet if and only if G contains an induced copy of P_4. Hence we have $w \leq \mathrm{FF}(w,N) \leq \mathrm{FFG}(w,P_4) = w$ and so $\mathrm{FF}(w,N) = w$. Kierstead, Penrice, and Trotter [14] proved that $\mathrm{FFG}(w,P_5)$ is bounded by a function of w, and a consequence of a theorem of Gyárfás and Lehel [8] is that $\mathrm{FFG}(w,P_6)$ is unbounded. As noted in [14], combining results in these two papers gives that, when T is a tree, $\mathrm{FFG}(w,T)$ is bounded if and only if T does not contain $P_2 + 2P_1$ as an induced subgraph, where $P_2 + 2P_1$ is the disjoint union of a copy of P_2 and two copies of P_1.

Let $\underline{r}$ denote the chain with r elements. The disjoint union of posets P and Q is denoted $P + Q$, with each element in P incomparable to every element in Q. An *interval order* is a poset whose elements are closed intervals with $[x_1,x_2] < [y_1,y_2]$ if and only if $x_2 < y_1$. Fishburn [7] proved that a poset P is an interval order if and only if P is $(\underline{2}+\underline{2})$-free. The problem of determining the performance of First-Fit on interval orders is still open, despite significant efforts by various different research groups over the years. Currently, the best known bounds are $(5 - o(1))w \leq \mathrm{FF}(w, \underline{2}+\underline{2}) \leq 8w$. The lower bound is due to Kierstead, D. Smith, and Trotter [11]. The upper bound is due to Brightwell, Kierstead, and Trotter (unpublished), and independently Narayanaswamy and Babu [16], who improved on the breakthrough column construction method due to Pemmaraju, Raman, and Varadarajan [17].

The interval orders are the $(\underline{2}+\underline{2})$-free posets; we obtain a larger class of posets by forbidding the disjoint union of longer chains. Bosek, Krawczyk, and Szczypka [5] showed that when $r \geq s$, $\mathrm{FF}(w, \underline{r} + \underline{s}) \leq (3r-2)(w-1)w + w$. Joret and Milans [10] improved the bound to $\mathrm{FF}(w, (\underline{r}+\underline{s})) \leq 8(r-1)(s-1)w$. Dujmović, Joret, and Wood [6] further improved the bound to $\mathrm{FF}(w, (\underline{r}+\underline{r})) \leq 8(2r-3)w$, which is best possible up to the constants.

The *ladder* of height n, denoted L_n, consists of two disjoint chains $x_1 < \cdots < x_n$ and $y_1 < \cdots < y_n$ with $x_i \leq y_j$ if and only if $i \leq j$ and no relations of the form $y_i \leq x_j$. Kierstead and M. Smith [12] showed that $\mathrm{FF}(w, L_2) = w^2$ and $\mathrm{FF}(2, L_n) \leq 2n$. They also proved the general bound $\mathrm{FF}(w, L_n) \leq w^{\gamma(\lg(w)+\lg(n))}$, where $\lg(x)$ denotes the base-2 logarithm; this result plays an important role in our main theorem.

1.2 Our Results

Our aim is to say something about the behavior of $\mathrm{FF}(w, Q)$ in terms of the structure of Q. We obtain subexponential bounds on $\mathrm{FF}(w, Q)$ when Q belongs to a particular family of posets $\mathcal{Q}$, and we also give an exponential lower bound on $\mathrm{FF}(w, Q)$ when $Q \notin \mathcal{Q}$. From the point of view of the First-Fit algorithm, efficiency is vastly improved if a single poset in $\mathcal{Q}$ is forbidden. From the point of view of an adversary, forcing First-Fit to use exponentially many chains requires all posets in $\mathcal{Q}$ to appear.

For each $x \in P$, we define the *above set* of x, denoted $A(x)$, to be $\{y \in P \colon y > x\}$; also, when S is a set of points, we define $A(S)$ to be $\bigcup_{x \in S} A(x)$. Similarly, the *below set* of x, denoted $B(x)$, is $\{y \in P \colon y < x\}$ and we extend this to sets via $B(S) = \bigcup_{x \in S} B(x)$. We define $A[x] = A(x) \cup \{x\}$ and similarly for $B[x]$. The *series composition* of posets $S_1, \ldots, S_n$, denoted $S_1 \otimes \cdots \otimes S_n$, produces a poset S which has disjoint copies of $S_1, \ldots, S_n$ arranged so that $x < y$ whenever $x \in S_i$, $y \in S_j$ and $i < j$. The *blocks* of S are the subposets $S_1, \ldots, S_n$.

2 Dichotomy Theorem

A poset is *ladder-like* if its elements can be partitioned into two chains C_1 and C_2 such that if $(x, y) \in C_1 \times C_2$ and x is comparable to y, then $x < y$. Our first lemma shows that every ladder-like poset is contained in a sufficiently large ladder.

Lemma 1. *If P is a ladder-like poset of size n, then P is a subposet of L_n.*

Proof. Let P be a ladder-like poset of size n. Clearly the 1-element poset is a subposet of L_1, and so we may assume $n \geq 2$. Let C_1 and C_2 be a chain partition of P such that whenever $(x, y) \in C_1 \times C_2$ and x and y are comparable, we have $x < y$. Suppose that P has a maximum element u. Recall that L_n consists of chains $x_1 < \cdots < x_n$ and $y_1 < \cdots < y_n$ with $x_i \leq y_j$ if and only if $i \leq j$. By induction, $P - u$ can be embedded into the copy of L_{n-1} in L_n induced by $\{x_1, \ldots, x_{n-1}\} \cup \{y_1, \ldots, y_{n-1}\}$. Allowing y_n to play the role of u completes a copy of P in L_n. Next, suppose that P has no maximum element. Let $u = \max C_2$, let $S = \{v \in C_1 \colon v \parallel u\}$, and let $s = |S|$. Since P has no maximum element, it follows that $s \geq 1$. By induction, $P - S$ can be embedded in the copy of L_{n-s} in L_n induced by

$\{x_1, \ldots, x_{n-s}\} \cup \{y_1, \ldots, y_{n-s}\}$. Allowing $\{x_{n-s+1}, \ldots, x_n\}$ to play the role of S completes a copy of P in L_n. $\square$

The performance of First-Fit on a poset P can be analyzed using a static structure. A *wall* of a poset P is an ordered chain partition $(C_1, \ldots, C_t)$ such that for each element $x \in C_j$ and each $i < j$, there exists $y \in C_i$ such that $y \parallel x$. It is clear that every ordered chain partition produced by First-Fit is a wall, and conversely, each wall W of P is output by First-Fit when the elements of P are presented in order according to W. Hence, the worst-case performance of First-Fit on P is equal to the maximum size of a wall in P. A *subwall* of a wall W is obtained from W by deleting zero or more of the chains in W. Note that if W is a wall of P, then each subwall of W is a wall of the corresponding subposet of P.

For each positive integer k, we construct a poset called the *reservoir* of width k, denoted R_k, and a corresponding wall W_k of size $2^k - 1$. The reservoirs provide an example of a family of posets which are good at avoiding subposets and yet still have exponential First-Fit performance.

Theorem 2. *For each $k \geq 1$, the reservoir R_k has width k and a wall W_k of size $2^k - 1$.*

Proof. Let R_1 be the 1-element poset, and let W_1 be the chain partition of R_1. For $k \geq 2$, we first construct R_k using R_{k-1} and W_{k-1}. Then, we give a presentation order for R_k which forces First-Fit to use at least $2^k - 1$ chains. Let $W_{k-1} = (C_1, \ldots, C_m)$ where $m = 2^{k-1} - 1$, and for $0 \leq i \leq m$, let $\hat{S}_i$ be the subwall $(C_1, \ldots, C_i)$ with corresponding subposet S_i. (Although S_0 and $\hat{S}_0$ are empty, they are convenient for describing R_k.) Let S be the series composition of disjoint copies of $S_m, S_{m-1}, \ldots, S_0$, and R_{k-1} in this order, so that $S = S_m \circledcirc S_{m-1} \circledcirc \cdots \circledcirc S_0 \circledcirc R_{k-1}$. The poset R_k consists of a copy of S and a chain X where $X = \{x_{m+1} < \cdots < x_1\}$ and each x_i satisfies $A(x_i) \cap S = \varnothing$ and $B(x_i) \cap S = S_i \cup \cdots \cup S_m$. See Figure 1.

Note that since S is a series composition of posets of width at most $k-1$, it follows that S has width at most $k-1$. Adding X increases the width by at most 1, and so R_k has width at most k. An antichain in the top copy of R_{k-1} of size $k-1$ and x_1 form an antichain in R_k of size k.

It remains to show that First-Fit might use as many as $2^k - 1$ chains to partition R_k. Consider the partial presentation order given by $\hat{S}_m, x_{m+1}, \hat{S}_{m-1}, x_m, \ldots, \hat{S}_1, x_2, \hat{S}_0, x_1$. We claim that First-Fit assigns color j to x_j for $1 \leq j \leq m+1$. Indeed, when $\hat{S}_{j-1}$ is presented, the points in S_{j-1} are above all previously presented points except $\{x_{j+1}, \ldots, x_{m+1}\}$, which have already been assigned colors larger than j. It follows that First-Fit uses colors $\{1, \ldots, j-1\}$ on S_{j-1}. Next, x_j is presented; since x_j is above all previously presented points except those in S_{j-1}, it follows that First-Fit assigns color j to x_j.

In the final stage, we present the top copy of R_{k-1} in order given by W_{k-1}. This copy of R_{k-1} is incomparable to each point in X and it follows that First-Fit uses m new colors on these points. In total, First-Fit uses $(m+1) + m$ colors, and $2m + 1 = 2^k - 1$. $\square$

If Q is a poset such that $\mathrm{FF}(w, Q)$ is subexponential in w, then Theorem 2 implies that Q is a subposet of a sufficiently large reservoir R_k. These posets have a nice description.

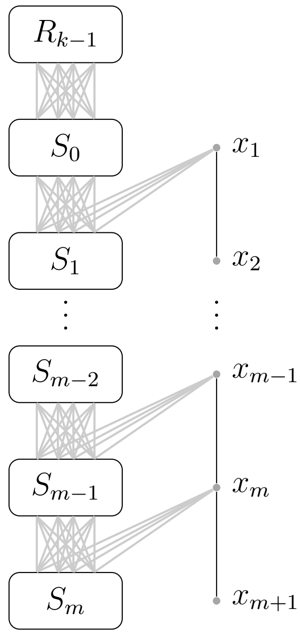

Figure 1: Reservoir Construction

Definition 3. *Let $\mathcal{Q}$ be the minimal poset family which contains the ladder-like posets and is closed under series composition.*

Our next lemma shows that $\mathcal{Q}$ characterizes the posets of width 2 that appear in reservoirs.

Lemma 4. *Let Q be a poset of width 2. Some reservoir R_k contains Q as a subposet if and only if $Q \in \mathcal{Q}$.*

Proof. If Q is ladder-like and has t elements, then Q is a subposet of L_t by Lemma 1, and L_t is a subposet of a sufficiently large reservoir. Suppose that $Q = Q_1 \oslash Q_2$ for some $Q_1, Q_2 \in \mathcal{Q}$ with $|Q_1|, |Q_2| < |Q|$. By induction, Q_1 and Q_2 are subposets of R_k for some k. Since R_{k+1} contains the series composition of two copies of R_k, it follows that Q is a subposet of R_{k+1}.

Let Q be a poset of width 2 that is contained in some reservoir. We show that $Q \in \mathcal{Q}$ by induction on $|Q|$. Let k be the least positive integer such that $Q \subseteq R_k$, and let $S_0, \ldots, S_m$, S, and X be as in the definition of R_k. If $Q \cap S$ is a chain, then $(Q \cap S, Q \cap X)$ is a chain partition witnessing that Q is ladder-like, and so $Q \in \mathcal{Q}$. Let y, z be a maximal incomparable pair in $Q \cap S$, meaning that if $y', z' \in Q \cap S$, $y' \geq y$, $z' \geq z$ and $(y', z') \neq (y, z)$, then y' and z' are comparable. We claim that if $u \in Q$ and u is above one of $\{y, z\}$, then u is above both y and z. This holds for $u \in Q \cap S$ by maximality of the pair y, z. This holds for $u \in Q \cap X$ since $y \parallel z$ implies that y and z belong to the same block in S, and all comparison relations between $u \in X$ and elements in S depend only on their block in S.

Since Q has width 2, it follows that $Q = Q_1 \oslash Q_2$ where $Q_1 = B[y] \cup B[z]$ and $Q_2 = A(y) \cup A(z)$. Unless Q_2 is empty and $Q_1 = Q$, it follows by induction that $Q_1, Q_2 \in \mathcal{Q}$ and therefore $Q \in \mathcal{Q}$ also. Suppose that no point in Q is above y or z. Since no point in X is below a point in S, it follows that $Q \cap X = \varnothing$, or else a point in $Q \cap X$ would complete an antichain of size 3 with $\{y, z\}$.

Therefore $Q \subseteq S$. Note that Q is not contained in one of the blocks in S by minimality of k since each such block is a subposet of R_{k-1}. It follows that $Q = Q_1 \otimes Q_2$ for posets Q_1 and Q_2 with $|Q_1|, |Q_2| < |Q|$. By induction, $Q_1, Q_2 \in \mathcal{Q}$ and so $Q \in \mathcal{Q}$ also. □

As a consequence of Lemma 4 and Theorem 2, it follows that $\mathrm{FF}(w, Q) \geq 2^w - 1$ when $Q \notin \mathcal{Q}$. It turns out that the performance of First-Fit is subexponential when $Q \in \mathcal{Q}$. Our next theorem shows how upper bounds on $\mathrm{FF}(w, Q_1)$ and $\mathrm{FF}(w, Q_2)$ can be used to obtain an upper bound on $\mathrm{FF}(w, Q_1 \otimes Q_2)$. A *Dilworth coloring* of a poset P of width w is a function $\varphi\colon P \to [w]$, where $[w] = \{1, \ldots, w\}$ such that the preimages of φ form a Dilworth partition.

Theorem 5. *Let Q_1 and Q_2 be posets, let w, s, and t be integers such that $\mathrm{FF}(w, Q_1) < s$ and $\mathrm{FF}(w, Q_2) < t$, and let $Q = Q_1 \otimes Q_2$. We have $\mathrm{FF}(w, Q) \leq stw^2 + (s+t)w$.*

Proof. For an ordered chain partition $\mathcal{C}$ of a poset P, an *ascending $\mathcal{C}$-chain* is a chain $x_1 < \cdots < x_k$ such that the chain in $\mathcal{C}$ containing x_i precedes the chain containing x_j for $i < j$. Similarly, a *descending $\mathcal{C}$-chain* is a chain $x_1 > \cdots > x_k$ such that the chain in $\mathcal{C}$ containing x_i precedes the chain containing x_j for $i < j$. The *$\mathcal{C}$-depth* of a point x, denoted $d_{\mathcal{C}}(x)$, is the size of a maximum ascending $\mathcal{C}$-chain with bottom element x and the *$\mathcal{C}$-height* of a point x, denoted $h_{\mathcal{C}}(x)$, is the size of a maximum descending $\mathcal{C}$-chain with top element x.

Let P be a Q-free poset of width at most w, and let $\mathcal{C}$ be a wall of P. We show that $|\mathcal{C}| \leq stw^2 + (s+t)w$. We claim that for each $x \in P$, at least one of the inequalities $h_{\mathcal{C}}(x) \leq s$, $d_{\mathcal{C}}(x) \leq t$ holds. Otherwise, if $h_{\mathcal{C}}(x) \geq s+1$ and $d_{\mathcal{C}}(x) \geq t+1$, then we obtain a copy of Q in P as follows. Let $x > y_1 > y_2 > \cdots > y_s$ be a descending $\mathcal{C}$-chain and let $x < z_1 < z_2 < \cdots < z_t$ be an ascending $\mathcal{C}$-chain. Let P_1 be the subposet of P consisting of all $u \in P$ such that for some y_i, the points u and y_i share a chain in $\mathcal{C}$ and $u \leq y_i$. Let $\mathcal{C}_1$ be the restriction of $\mathcal{C}$ to P_1 and observe that $\mathcal{C}_1$ is a wall of P_1. Indeed, suppose that $C, C' \in \mathcal{C}_1$ where C precedes C', and let $(y_i, y_j) = (\max C, \max C')$. Let $v \in C'$ and note that v and y_j share a chain in $\mathcal{C}$. Let u be a point in P such that u belongs to the same chain in $\mathcal{C}$ as y_i and $u \parallel v$. Note that $u \leq y_i$, since otherwise $u > y_i > y_j \geq v$, contradicting $u \parallel v$. Therefore $u \in P_1$ and $u \in C$. Since $\mathcal{C}_1$ is a wall of P_1 of size s and $s > \mathrm{FF}(w, Q_1)$, it follows that P_1 contains a copy of Q_1. Similarly, we let P_2 be the subposet of P consisting of all $u \in P$ such that for some z_i, the points u and z_i share a chain in $\mathcal{C}$ and $u \geq z_i$. Restricting $\mathcal{C}$ to P_2 gives a wall $\mathcal{C}_2$ of size t analogously, and since $t > \mathrm{FF}(w, Q_2)$, it follows that P_2 contains a copy of Q_2. Since every element in P_1 is less than x and x is less than every element in P_2, it follows that P contains a copy of Q.

The *lower part* of P, denoted by L, is $\{x \in P\colon h_{\mathcal{C}}(x) \leq s\}$ and the *upper part* of P, denoted by U, is $P - L$. Note that $\{L, U\}$ is a partition of P, that $h_{\mathcal{C}}(x) \leq s$ for $x \in L$, and that $d_{\mathcal{C}}(x) \leq t$ for $x \in U$. Let $\mathcal{C}_U$ be the subwall of $\mathcal{C}$ consisting of all chains that are contained in U, and let $\mathcal{C}_{U,j}$ be the subwall of $\mathcal{C}_U$ consisting of the chains $C \in \mathcal{C}_U$ such that

$d_{\mathcal{C}}(\min C) = j$. We claim that the minimum elements of the chains in $\mathcal{C}_{U,j}$ form an antichain. Suppose that $C, C' \in \mathcal{C}_{U,j}$ and that C precedes C'. Since C precedes C', it is not possible for $\min C > \min C'$. Therefore if $\min C$ and $\min C'$ are comparable, then it must be that $\min C < \min C'$, and it would follow that $d_{\mathcal{C}}(\min C) > d_{\mathcal{C}}(\min C')$. Hence $|\mathcal{C}_{U,j}| \leq w$ for $1 \leq j \leq t$ and so $|\mathcal{C}_U| \leq tw$. A symmetric argument shows that the sublist $\mathcal{C}_L$ consisting of all chains that are contained in L satisfies $|\mathcal{C}_L| \leq sw$.

It remains to bound the number of chains in $\mathcal{C}$ that contain points in both U and L. Let $\mathcal{C}_{LU}$ be the sublist of $\mathcal{C}$ consisting of these chains. Note that for each $C \in \mathcal{C}$, we have that $y, z \in C$ and $y < z$ implies that $h_{\mathcal{C}}(y) \leq h_{\mathcal{C}}(z)$ and $d_{\mathcal{C}}(y) \geq d_{\mathcal{C}}(z)$. It follows that each point in $C \cap L$ is less than each point in $C \cap U$. Let $\varphi\colon P \to [w]$ be a Dilworth coloring. For each $C \in \mathcal{C}_{LU}$ with $y = \max(C \cap L)$ and $z = \min(C \cap U)$, we assign to C the *signature* $(\varphi(y), h_{\mathcal{C}}(y), \varphi(z), d_{\mathcal{C}}(z))$. We claim that the signatures are distinct. Suppose that $C, C' \in \mathcal{C}_{LU}$ have the same signature and that C precedes C'. Let $y = \max(C \cap L)$, $z = \min(C \cap U)$, $y' = \max(C' \cap L)$, and $z' = \min(C' \cap U)$. Note that $y < z$ is a cover relation in C and $y' < z'$ is a cover relation in C'. Since $\varphi(y) = \varphi(y')$, it follows that y and y' are comparable. Since $h_{\mathcal{C}}(y) = h_{\mathcal{C}}(y')$, it must be that $y < y'$. Since $\varphi(z) = \varphi(z')$, it follows that z' and z are comparable. Since $d_{\mathcal{C}}(z') = d_{\mathcal{C}}(z)$, it must be that $z' < z$. We now have that $y < z$ is a cover relation in C but $y < y' < z' < z$ for points z', y' that appear in a chain C' that follows C, contradicting that $\mathcal{C}$ is a wall.

Since the assigned signatures are distinct, we have that $|\mathcal{C}_{LU}| \leq stw^2$. It follows that $|\mathcal{C}| \leq |\mathcal{C}_{LU}| + |\mathcal{C}_L| + |\mathcal{C}_U| \leq stw^2 + sw + tw$. □

Corollary 6. *Let $Q = Q_1 \otimes \cdots \otimes Q_k$. If $\mathrm{FF}(w, Q_i) \leq 2^{c_i (\lg w)^2}$ for $1 \leq i \leq k$, then $\mathrm{FF}(w, Q) \leq 2^{(c+6k)(\lg w)^2}$, where $c = \sum_{i=1}^{k} c_i$.*

Proof. By induction on k. For $k = 1$, the claim is clear. Suppose $k \geq 2$. Since $\mathrm{FF}(1, Q) \leq 1$, we may assume $w \geq 2$. Let $R = Q_1 \otimes \cdots \otimes Q_{k-1}$. By induction, $\mathrm{FF}(w, R) \leq 2^{(c'+6(k-1))(\lg w)^2}$, where $c' = \sum_{i=1}^{k-1} c_i$. By Theorem 5 with $s \leq 1 + 2^{(c'+6(k-1))(\lg w)^2}$ and $t \leq 1 + 2^{c_k (\lg w)^2}$, we have $\mathrm{FF}(w, Q) \leq stw^2 + (s+t)w \leq 3stw^2 < 2^2 \cdot 2^{(c'+6(k-1))(\lg w)^2+1} \cdot 2^{c_k(\lg w)^2+1} \cdot 2^{2\lg w}$. It follows that $\lg[\mathrm{FF}(w, Q)] < (c' + c_k + 6(k-1))(\lg w)^2 + 4 + 2\lg w \leq (c+6k)(\lg w)^2$. □

The following key result due to Kierstead and M. Smith [12] shows that First-Fit uses a subexponential number of chains on ladder-free posets. We follow with the characterization of posets Q for which $\mathrm{FF}(w, Q)$ is subexponential.

Theorem 7 (Kierstead–M. Smith [12])**.** *For some constant γ, we have $\mathrm{FF}(w, L_n) \leq w^{\gamma(\lg(w)+\lg(n))}$.*

Theorem 8 (Dichotomy Theorem)**.** *Let Q be an n-element poset of width 2. If $Q \in \mathcal{Q}$, then there exists a constant C (depending only on Q) such that $\mathrm{FF}(w, Q) \leq 2^{C(\lg w)^2}$; in fact, $C = O(n)$ suffices. If $Q \notin \mathcal{Q}$, then $\mathrm{FF}(w, Q) \geq 2^w - 1$.*

Proof. Suppose $Q \notin \mathcal{Q}$. By Theorem 2 and Lemma 4, we have $\mathrm{FF}(w, Q) \geq 2^w - 1$. Suppose that $Q \in \mathcal{Q}$. Since $\mathrm{FF}(1, Q) \leq 1$, we may assume $w \geq 2$. Since $Q \in \mathcal{Q}$, it follows that $Q = Q_1 \otimes \cdots \otimes Q_k$ for some ladder-like posets $Q_1, \ldots, Q_k$. For $1 \leq i \leq k$, let $n_i = |Q_i|$. Since Q_i is ladder-like, Theorem 7 implies that $\mathrm{FF}(w, Q_i) \leq 2^{c_i(\lg w)^2}$ where $c_i = \gamma(1 + \frac{\lg(n_i)}{\lg(w)}) \leq \gamma(1+$

$\lg n_i$). By Corollary 6, it follows that $\mathrm{FF}(w,Q) \le 2^{(c+6k)(\lg w)^2}$, where $c = \sum_{i=1}^{k} c_i$. Hence, it suffices to take $C = 6k + c = 6k + \sum_{i=1}^{k} c_i \le (6+\gamma)k + \gamma \sum_{i=1}^{k} \lg n_i$. Since $\sum_{i=1}^{k} n_i = n$, it follows by convexity that $\sum_{i=1}^{k} \lg n_i \le k \lg(n/k) \le (n/e)\lg e$, where e is the base of the natural logarithm. Using $k \le n$, we conclude $C \le (6+\gamma)n + \gamma(n/e)\lg e = O(n)$. □

Theorem 8 provides a large separation in the behavior of First-Fit on Q-free posets according to whether or not $Q \in \mathcal{Q}$. It may be that even stronger results are possible. Theorem 5 shows that if $\mathrm{FF}(w,Q_1)$ and $\mathrm{FF}(w,Q_2)$ are polynomial in w, then so is $\mathrm{FF}(w,Q_1 \otimes Q_2)$. For large n, the best known lower bound on $\mathrm{FF}(w,L_n)$ is $w^{\lg(n-1)}/(n-1)$, due to Bosek, Kierstead, Krawczyk, Matecki, and M. Smith [3]. This leaves open the possibility that $\mathrm{FF}(w,L_n)$ is polynomial in w for each fixed n. If so, then the separation provided by the Dichotomy Theorem would improve, yielding that $\mathrm{FF}(w,Q)$ is polynomial when $Q \in \mathcal{Q}$ and exponential when $Q \notin \mathcal{Q}$.

Question 9. *Is it true for each fixed n that $\mathrm{FF}(w,L_n)$ is bounded by a polynomial in w?*

It is clear that $\mathrm{FF}(w,L_1) = w$ and Kierstead and M. Smith [12] proved that $\mathrm{FF}(w,L_2) = w^2$. Note that $L_3 = Q_1 \otimes Q_2 \otimes Q_3$ where Q_1 and Q_3 are 1-element posets and Q_2 is the N poset. Since $\mathrm{FF}(w,Q_1) = \mathrm{FF}(w,Q_3) = 0$ and $\mathrm{FF}(w,Q_2) = w$, it follows from Theorem 5 that $\mathrm{FF}(w,L_3)$ is polynomial in w. A more careful analysis, along the lines of Kierstead and M. Smith's proof of $\mathrm{FF}(w,L_2) = w^2$, shows that $\mathrm{FF}(w,L_3) \le w^2(w+1)$. Question 9 is open for $n \ge 4$.

It would also be interesting to better understand the behavior of First-Fit on Q-free posets when $Q \notin \mathcal{Q}$. The smallest poset of width 2 that is not in $\mathcal{Q}$ is the *skewed butterfly*, denoted $\hat{B}$, which consists of the chains $x_1 < x_2 < x_3$ and $y_1 < y_2$ with relations $x_1 < y_2$ and $y_1 < x_3$. What is $\mathrm{FF}(w,\hat{B})$?

3 First-Fit on Butterfly-Free Posets

The *butterfly poset*, denoted B, is $Q \otimes Q$, where Q is the 2-element antichain. In this section, we obtain the asymptotics of $\mathrm{FF}(w,B)$. The performance of First-Fit on butterfly-free posets is strongly related to the bipartite Turán number for C_4. Kövari, Sós, Turán [15] showed that the maximum number of edges in a subgraph of $K_{n,n}$ that excludes C_4 is $(1+o(1))n^{3/2}$.

Lemma 10 (Kövari–Sós–Turán [15]). *Let q be a prime power, and let $n = q^2+q+1$. There exists a $(q+1)$-regular spanning subgraph of $K_{n,n}$ that has no 4-cycle.*

We also need a standard result about the density of primes.

Theorem 11 (Hoheisel [9]). *There exists a real number θ with $\theta < 1$ such that for all sufficiently large real numbers x, there is a prime in the interval $[x - x^{\theta}, x]$.*

Since the result of Hoheisel [9], many research groups have improved the bound on θ; see Baker and Harman [1] for the history. The current best bound is $\theta = 0.525$, due to Baker, Harman, and Pintz [2].

Theorem 12. $\mathrm{FF}(w, B) \geq (1 - o(1))w^{3/2}$.

Proof. By Theorem 11 and standard asymptotic arguments, we may assume that w has the form q^2+q+1, where q is prime. By Lemma 10, there exists a $(q+1)$-regular (X, Y)-bigraph G with parts of size w that has no 4-cycle. Since G is a regular bipartite graph, it follows from Hall's Theorem that G has a perfect matching M. Let $G' = G - M$, and let L be an ordering of $E(G')$.

Using G', we construct a B-free poset P of width w and a wall of P size $|E(G)|$. It will then follow that $\mathrm{FF}(w, B) \geq |E(G)| = (q + 1)w = (1 - o(1))w^{3/2}$. Let I_X be the set of all pairs (x, e) such that $x \in X$, $e \in E(G')$, and e is incident to x. Similarly, let I_Y be the set of all pairs (y, e) such that $y \in Y$, $e \in E(G')$ and e is incident to y. We construct P so that M is a maximum antichain, $B(M) = I_X$, and $A(M) = I_Y$. The subposet induced by $I_X \cup M$ consists of w incomparable chains, indexed by M. For $x_iy_i \in M$ with $x_i \in X$ and $y_i \in Y$, the chain associated with x_iy_i consists of all pairs $(x_i, e) \in I_X$ in order according to L followed by top element x_iy_i. The subposet induced by $M \cup I_Y$ also consists of w incomparable chains, indexed by M. For $x_iy_i \in M$ with $x_i \in X$ and $y_i \in Y$, the chain associated with x_iy_i in the subposet induced by $M \cup I_y$ consists of bottom element x_iy_i followed by all pairs $(y_i, e) \in I_Y$ in reverse order according to L. Note that if e is the first edge in L and $e = xy$, then (x, e) is minimal in P and (y, e) is maximal. The chains in $I_X \cup M$ and the chains in $M \cup I_Y$ combine to form a Dilworth partition of P of size w; let D_i be the Dilworth chain containing x_iy_i. It remains to describe the relations between points in I_X and points in I_Y. For $(x, e_1) \in I_X$ and $(y, e_2) \in I_Y$, we have that (x, e_1) is covered by (y, e_2) if and only if $e_1 = e_2 = xy \in E(G')$.

We claim that P is B-free. For each element $z \in I_X \cup M$, we have that $B(z)$ is a chain. Hence, a maximal element in a copy of B must belong to I_Y. Similarly, since $A(z)$ is a chain when $z \in M \cup I_Y$, a minimal element in a copy of B must belong to I_X. In a chain of cover relations from $(x, e_1) \in I_X$ up to $(y, e_2) \in I_Y$, either all points stay in the same Dilworth chain D_i, implying that $xy = x_iy_i \in M$, or there is a cover relation from a point in D_i to a point in D_j, that implying $xy = x_iy_j$ with $x_iy_j \in E(G')$. In both cases, $(x, e_1) \leq (y, e_2)$ implies that $xy \in E(G)$, and it follows that a copy of B in P corresponds to a 4-cycle in G, a contradiction.

It remains to construct a wall W of P of size $|E(G)|$. The wall contains $|E(G')|$ chains of size 2 arranged in order according to L, followed by w singleton chains. For $e \in L$ with $e = xy$, the corresponding chain in the wall is $(x, e) < (y, e)$. These chains are followed by w singleton chains, each consisting of a point in M. Let C_i and C_j be chains in W with $i < j$, and let $z \in C_j$. We show that z is incomparable to some point in C_i. Since M is an antichain, we may assume that C_i is a chain of the form $(x, e) < (y, e)$. If C_j is a singleton chain containing only z, then z is incomparable to every element in P outside its Dilworth chain. Since (x, e) and (y, e) are in distinct Dilworth chains, it follows that C_i contains a point incomparable to z. Otherwise, C_j has the form $(x', e') < (y', e')$, and since $i < j$, it follows that e precedes e' in L. Suppose that $z = (x', e')$. If $(x', e') \parallel (x, e)$, then (x, e) is the desired point in C_i. Otherwise, (x', e') is comparable to (x, e), implying that (x, e) and (x', e') are in the same Dilworth chain and $x = x'$. Since e precedes e' in L, we have $(x, e) < (x', e')$. If (x', e') is also comparable to (y, e), it must be that $(x', e') < (y, e)$.

But now $(x, e) < (x', e') < (y, e)$ contradicts that (y, e) covers (x, e) in P. The case that $z = (y', e')$ is analogous. $\square$

In a poset P with a set of elements S, an *extremal point* of S is a minimal or maximal element in S.

Lemma 13. *Let C and D be chains in P. If $\min C \parallel \max D$ and $\max C \parallel \min D$, then C and D are pairwise incomparable. Consequently if C' and D' are chains and $(x_1, y_1), (x_2, y_2) \in C' \times D'$ are incomparable pairs, then $\min\{x_1, x_2\} \parallel \min\{y_1, y_2\}$ and $\max\{x_1, x_2\} \parallel \max\{y_1, y_2\}$.*

Proof. If $u \le v$, $u \in C$, and $v \in D$, then $\min C \le u \le v \le \max D$. If $u \le v$, $u \in D$, and $v \in C$, then $\min D \le u \le v \le \max C$. For the second part, either the statement is trivial or we apply the first part to the subchains of C' and D' with extremal points $\{x_1, x_2\}$ and $\{y_1, y_2\}$ respectively. $\square$

Starting with an arbitrary chain partition $\mathcal{C}$, iteratively moving elements to earlier chains produces a wall W with $|W| \le |\mathcal{C}|$. Beginning with a Dilworth partition, it follows that each poset P of width w has a *Dilworth wall* consisting of w chains. If R and S are sets of points in P, we write $R < S$ if $u < v$ when $(u, v) \in R \times S$.

Theorem 14. $\mathrm{FF}(w, B) \le (1 + o(1))w^{3/2}$.

Proof. Let P be a B-free poset and let $\mathcal{D}$ be Dilworth wall of P with $\mathcal{D} = (D_1, \ldots, D_w)$. Let R be the set of points $x \in P$ such that $A(x)$ is a chain. Let $R' = P - R$, and note that $B(x)$ is a chain for each $x \in R'$ since P is B-free.

Let $\mathcal{C}$ be a wall of P with $\mathcal{C} = (C_1, \ldots, C_t)$; we bound $|\mathcal{C}|$. Since $|\mathcal{D}| = w$, at most $2w$ chains in $\mathcal{C}$ contain an extremal point from a chain in $\mathcal{D}$. Also, no two chains in $\mathcal{C}$ are contained in the same chain in $\mathcal{D}$, and so at most w chains in $\mathcal{C}$ are contained in a chain in $\mathcal{D}$. Let $\mathcal{C}'$ be the subwall of $\mathcal{C}$ consisting of all chains $C \in \mathcal{C}$ that do not contain an extremal point of a chain in $\mathcal{D}$ but contain points from at least two chains in $\mathcal{D}$. We have that $|\mathcal{C}| \le |\mathcal{C}'| + 3w$. We claim that for each chain $C_i \in \mathcal{C}'$, we have that $C_i \cap R$ is contained in a chain in $\mathcal{D}$. Suppose that $C_i \cap R$ contains elements from at least two chains in $\mathcal{D}$. Let D_α be the Dilworth chain containing $\max C_i$, let $x = \max(C_i - D_\alpha)$, and let D_β be the Dilworth chain containing x. Let $m = \max D_\beta$, and note that $C_i \in \mathcal{C}'$ implies $m \notin C_i$. It follows that $m \in C_j$ for some $C_j \in \mathcal{C}$ with $j \ne i$; since $A(x)$ is a chain and $m > x$, it follows that m is comparable to every element in C_i and therefore $j < i$. Let y be the element covering x in C_i. Note that $y \in D_\alpha$ and y is comparable to everything in D_β since $A(x)$ is a chain, and this implies $\alpha < \beta$. Since $m, y \in A(x)$ and $A(x)$ is a chain, either $m < y$ or $m > y$. If $m > y$, then m is comparable to everything in D_α, contradicting $m \in D_\beta$ and $\alpha < \beta$. Similarly, if $m < y$, then y is comparable to every element in C_j, contradicting $y \in C_i$ and $j < i$. Therefore $C_i \cap R$ is contained in a single chain in $\mathcal{D}$. By a symmetric argument, $C_i \cap R'$ is contained in a single chain in $\mathcal{D}$.

It remains to bound $|\mathcal{C}'|$. Note that for each $C \in \mathcal{C}'$, we have that $C \cap R$ is contained in some Dilworth chain $D_\alpha \in \mathcal{D}$ and $C \cap R'$ is contained in some Dilworth chain $D_\gamma \in \mathcal{D}$, with $\alpha \ne \gamma$; we say that (α, γ) is the *signature* of $C \in \mathcal{C}'$ if $C \cap R \subseteq D_\alpha$ and $C \cap R' \subseteq D_\gamma$. Note that if $C_i, C_j \in \mathcal{C}'$ with $i < j$, then it is not possible for both C_i and C_j to have the same signature (α, γ), or else $C_i \cap R' < C_j < C_i \cap R$. Since the signatures are distinct, it follows that $|\mathcal{C}'| \le w^2$ and so $\mathrm{FF}(w, B) \le (1 + o(1))w^2$.

Let X and Y be disjoint copies of $\mathcal{D}$, and let G be the (X, Y)-bigraph in which $D_\alpha \in X$ and $D_\gamma \in Y$ are adjacent if and only if some chain in $\mathcal{C}'$ has signature (α, γ). We claim that G has no 4-cycle, implying $|\mathcal{C}'| = |E(G)| \le (1 + o(1))w^{3/2}$.

Suppose for a contradiction that G has a 4-cycle on $D_\alpha, D_\beta \in X$ and $D_\gamma, D_\delta \in Y$. Let C_i, C_j, C_k, C_ℓ be chains in $\mathcal{C}'$ with signatures (α, γ), (α, δ), (β, γ), and (β, δ), respectively. Assume, without loss of generality, that C_i precedes C_j in $\mathcal{C}$, and let $y_1 \in C_j \cap R' \subseteq D_\delta$. Since y_1 is in a later chain, it must be that $x_1 \parallel y_1$ for some $x_1 \in C_i$. Since $C_j \cap R$ and $C_i \cap R$ are both contained in D_α and $y_1 \in C_j \cap R' < C_j \cap R < C_i \cap R$, it follows that $x_1 \in C_i \cap R' \subseteq D_\gamma$. Therefore there is an incomparable pair $(x_1, y_1) \in (C_i \cap R') \times (C_j \cap R')$. A similar argument applied to C_k and C_ℓ with top parts in D_β shows that there is an incomparable pair $(x_2, y_2) \in (C_k \cap R') \times (C_\ell \cap R')$. Since $C_i \cap R', C_k \cap R' \subseteq D_\gamma$ and $C_j \cap R', C_\ell \cap R' \subseteq D_\delta$, it follows from Lemma 13 that there is an incomparable pair $(x, y) \in D_\gamma \times D_\delta$ with $x \le \min\{\max C_i \cap R', \max C_k \cap R'\}$ and $y \le \min\{\max C_j \cap R', \max C_\ell \cap R'\}$. Similarly, there is an incomparable pair $(x', y') \in D_\alpha \times D_\beta$ with $x' \ge \max\{\min C_i \cap R, \min C_j \cap R\}$ and $y' \ge \max\{\min C_k \cap R, \min C_\ell \cap R\}$. Since $x, y < x', y'$, it follows that $\{x, y, x', y'\}$ induces a copy of B in P.

Since $|\mathcal{C}| \le |\mathcal{C}'| + 3w \le (1 + o(1))w^{3/2}$, the bound on $FF(w, B)$ follows. □

Corollary 15. $\mathrm{FF}(w, B) = (1 + o(1))w^{3/2}$.

The *stacked butterfly* of height t, denoted B_t, is $Q_1 \otimes \cdots \otimes Q_t$, where each Q_i is a 2-element antichain. Note that B_{2k} is the series composition of k copies of B. A consequence of our results is that $\mathrm{FF}(w, B_t)$ is bounded by a polynomial in w for each fixed t.

Corollary 16. $\mathrm{FF}(w, B_{2k}) \le (1 + o(1))w^{3.5k-2}$

Proof. From Theorem 5 and Corollary 15 we have that

$$\mathrm{FF}(w, B_{2k}) \le (1 + o(1))w^2 \mathrm{FF}(w, B_{2(k-1)}) \mathrm{FF}(w, B_2) = (1 + o(1))w^{3.5k-2}.$$

□

It would be interesting to find lower bounds on $\mathrm{FF}(w, B_{2k})$. In particular, is $\mathrm{FF}(w, B_{2k})$ bounded below by a polynomial in w whose degree grows linearly in k?

4 Conclusions and Open Problems

A consequence of Theorem 8 is that $\mathcal{Q}$ is the family of posets Q such that $\mathrm{FF}(w, Q)$ is subexponential in w. It may be that $\mathcal{Q}$ is also the family of posets Q such that $\mathrm{FF}(w, Q)$ is polynomial in w. This is the case if and only if Question 9 has a positive answer. Alternatively, if Question 9 has a negative answer, then it would be interesting to understand what structural properties of Q lead to polynomial behavior of $\mathrm{FF}(w, Q)$.

Problem 17. *Characterize the posets Q for which $\mathrm{FF}(w, Q)$ is bounded above by a polynomial in w.*

We have focused on upper bounds for posets in $\mathcal{Q}$ and lower bounds for posets outside $\mathcal{Q}$. It would be nice to obtain better bounds for posets outside $\mathcal{Q}$. The smallest poset of width 2 that is outside $\mathcal{Q}$ is the *skewed butterfly* $\hat{B}$ consisting of disjoint chains $x_1 < x_2 < x_3$ and $y_1 < y_2$ with the cover relations $x_1 < y_2$ and $y_1 < x_3$. According to Theorem 2, we have $\mathrm{FF}(w, \hat{B}) \geq 2^w - 1$. What is $\mathrm{FF}(w, \hat{B})$? Although Bosek, Krawczyk, and Matecki [4] provide tower-type upper bounds on $\mathrm{FF}(w, Q)$, there may be room for significant improvement.

Question 18. *Is there any poset Q of width 2 for which $\mathrm{FF}(w, Q)$ is superexponential?*

We have studied the behavior of First-Fit on families that forbid a single poset Q, but it is also natural to ask about families that forbid a set of posets. If $\mathcal{S}$ is a set of posets, we say that a poset P is $\mathcal{S}$-free if no poset in $\mathcal{S}$ is a subposet of P. Let $\mathrm{FF}(w, \mathcal{S})$ be the maximum number of chains that First-Fit uses on an $\mathcal{S}$-free poset of width w.

Problem 19. *Characterize the sets $\mathcal{S}$ for which $\mathrm{FF}(w, \mathcal{S})$ is bounded by a polynomial in w.*

If $\mathcal{P}$ is a poset family that is closed under taking subposets, then $\mathcal{P}$ is exactly the set of posets that is $\mathcal{S}$-free, where $\mathcal{S}$ is the set of minimal posets not in $\mathcal{P}$. A solution to Problem 19 is therefore equivalent to a characterization of all subposet-closed families $\mathcal{P}$ such that First-Fit has polynomial behavior when restricted to $\mathcal{P}$. We suspect that this is a challenging problem, but the restriction of Problem 19 to $|\mathcal{S}| \leq 2$ is likely more accessible and even partial progress would still be interesting.

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