First-Fit chain partitions in partially ordered sets

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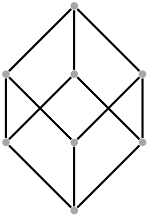
Chain Partitions



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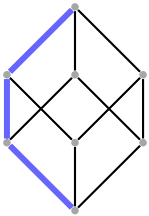

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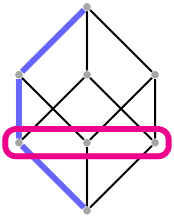

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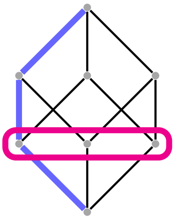

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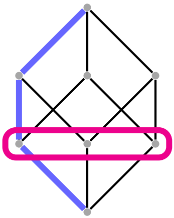

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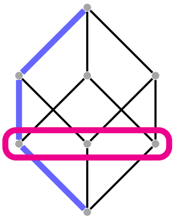

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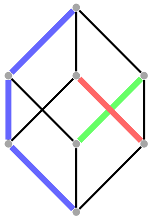

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- In terms of the width w, how many chains can First-Fit use?

Kierstead's First-Fit Example

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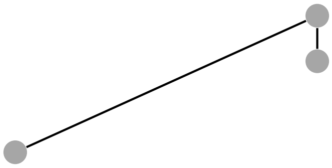

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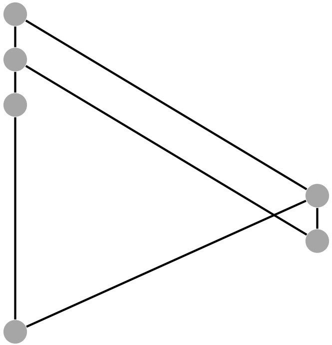

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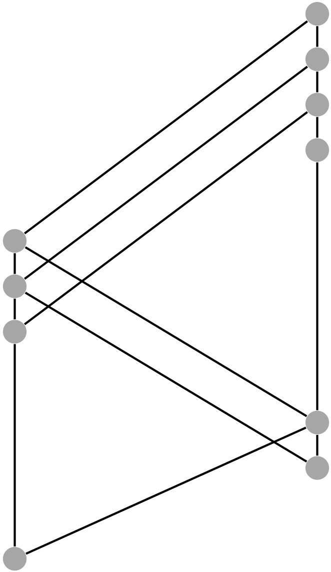

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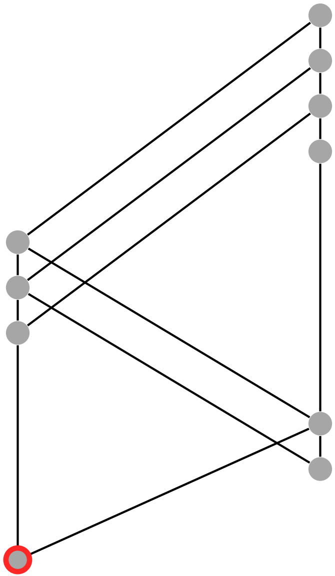

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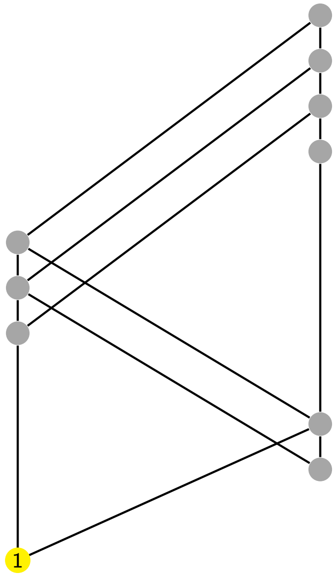

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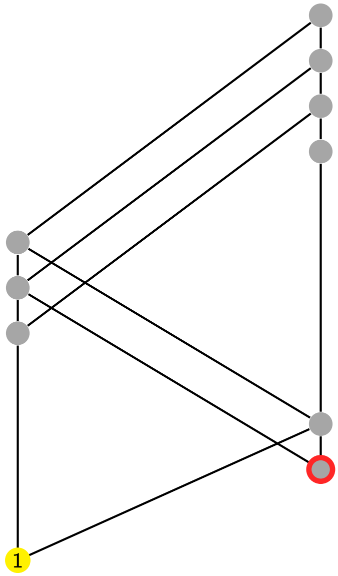

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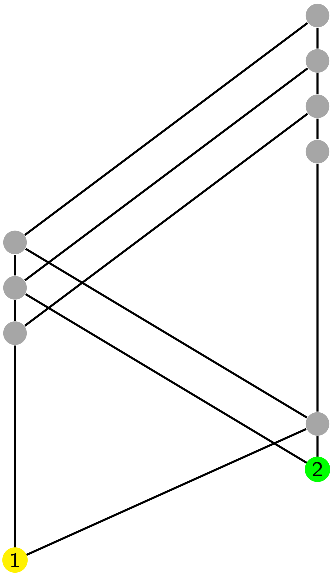

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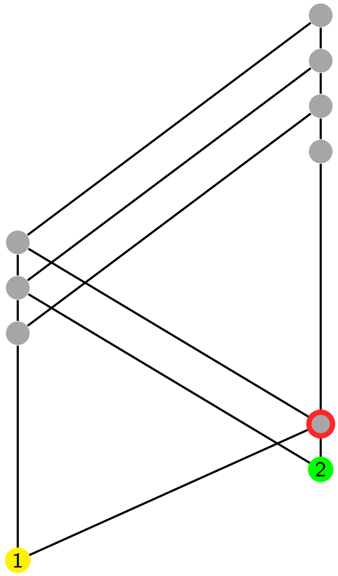

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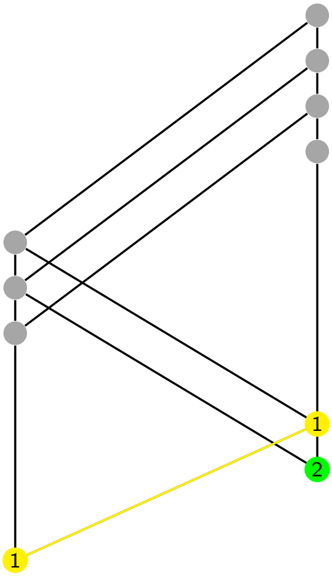

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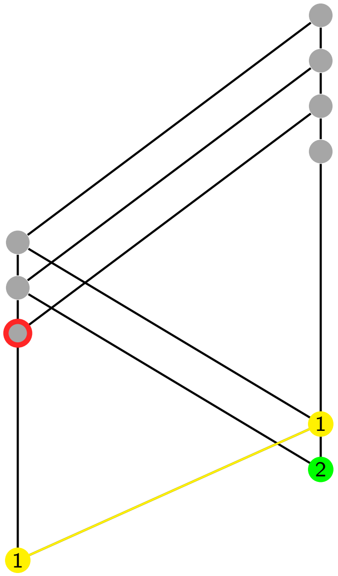

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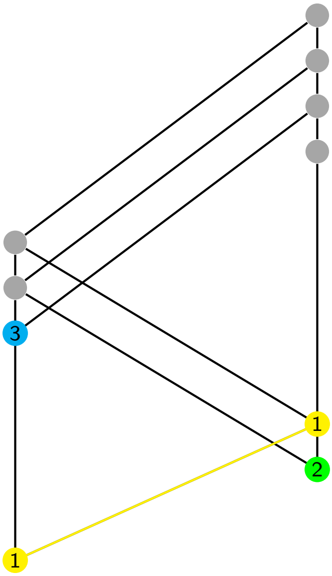

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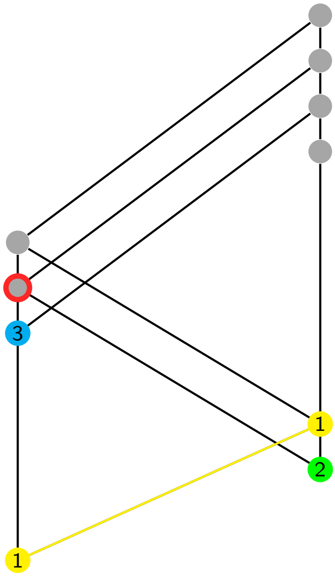

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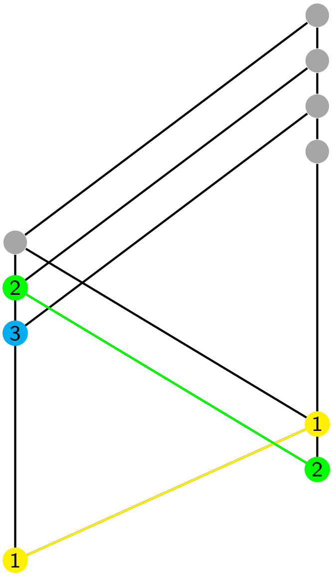

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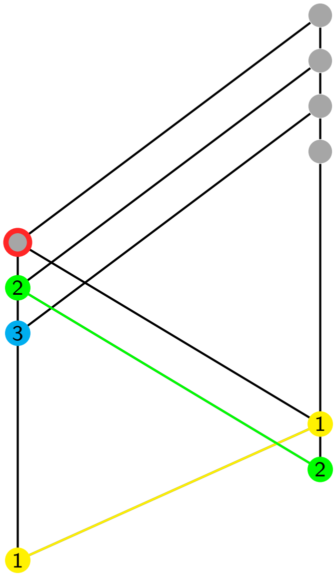

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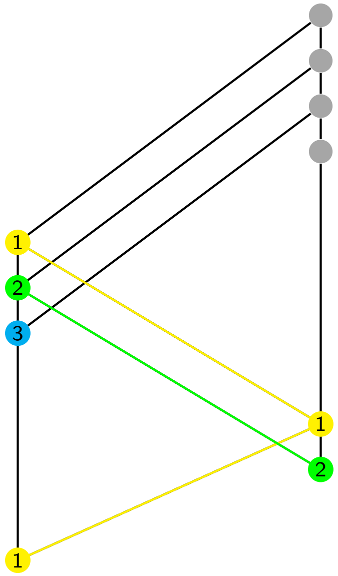

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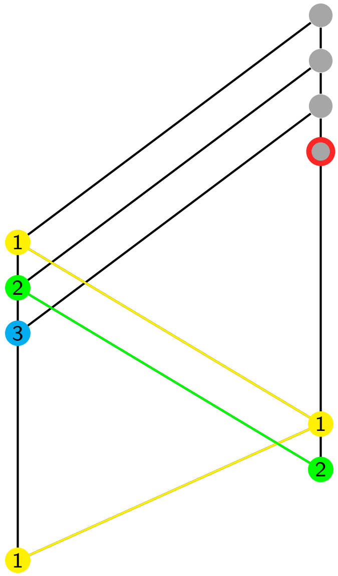

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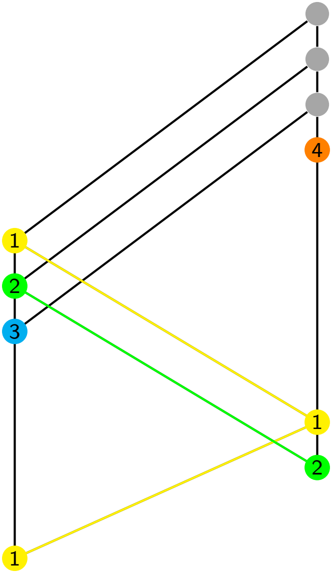

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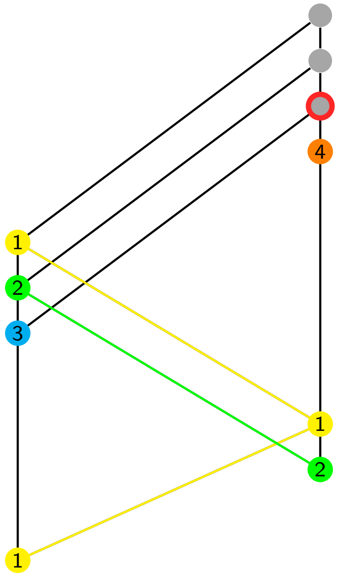

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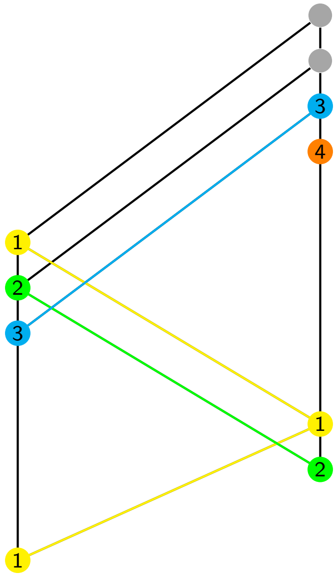

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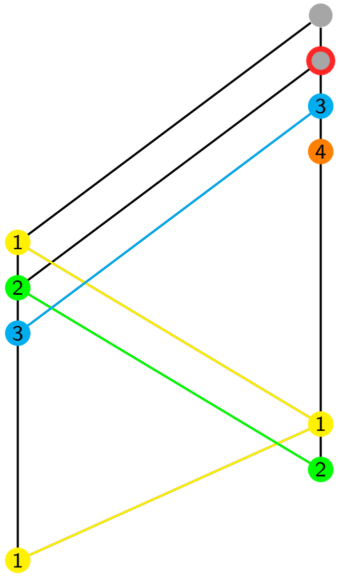

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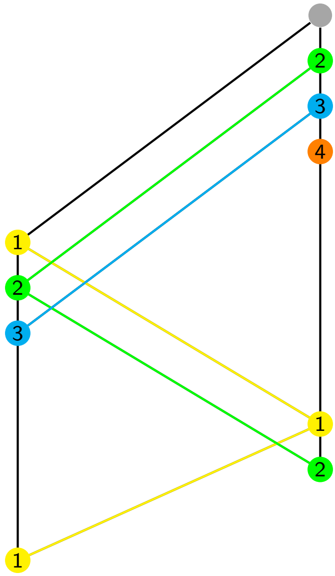

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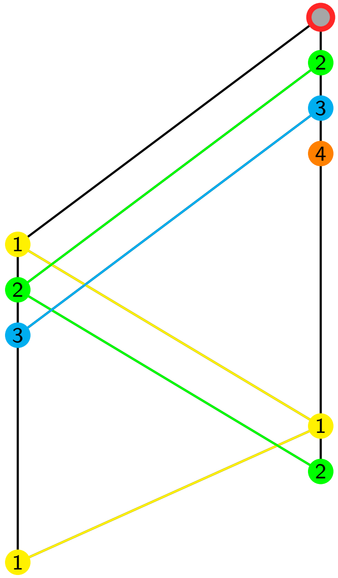

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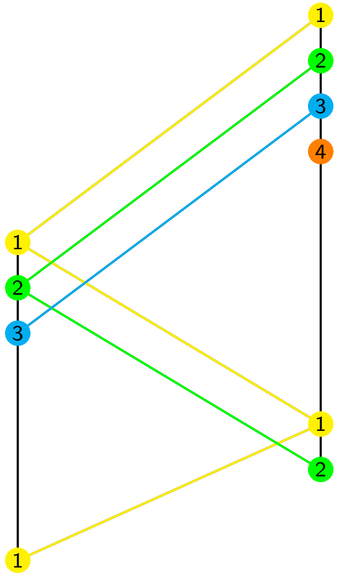

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Theorem (Bosek–Krawczyk–Matecki (2011))

If Q has width at most 2, then $\mathrm{FF}(w, Q)$ is bounded.

Prior Work

$\underline{4}$

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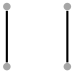

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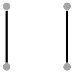

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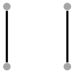

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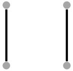

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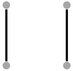

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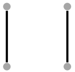

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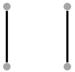

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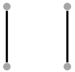

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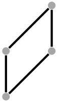
Main Question

What properties of Q determine the behavior of $\mathrm{FF}(w, Q)$?

Warmup: $\mathrm{FF}(w, L_2) \le w^2$ (Kierstead–M. Smith (2013))



L_2

- A *P*-wall is an ordered chain partition $(W_1, \ldots, W_t)$ of P produced by First-Fit.

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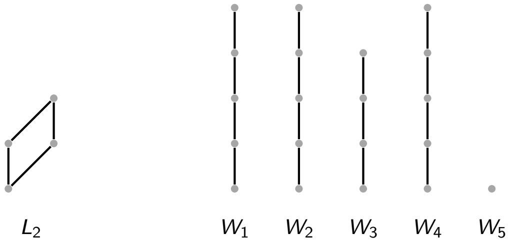

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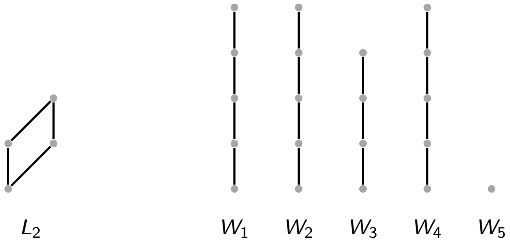

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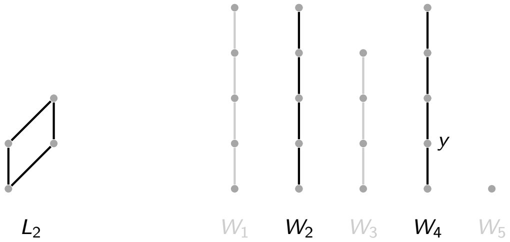

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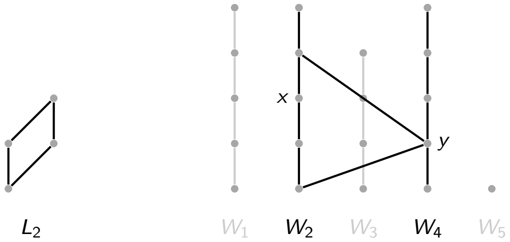

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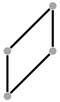

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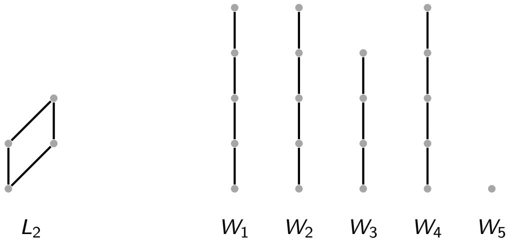

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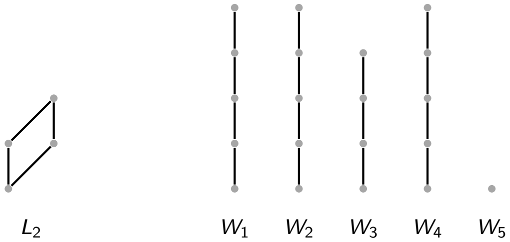
Warmup: $\mathrm{FF}(w, L_2) \leq w^2$ (Kierstead–M. Smith (2013))



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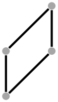
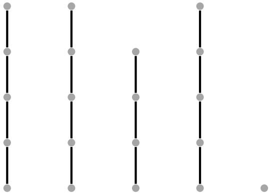


L_2



W_1 W_2 W_3 W_4 W_5

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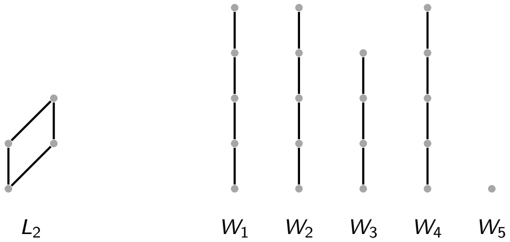

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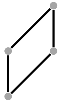

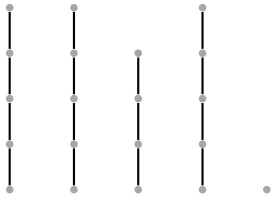

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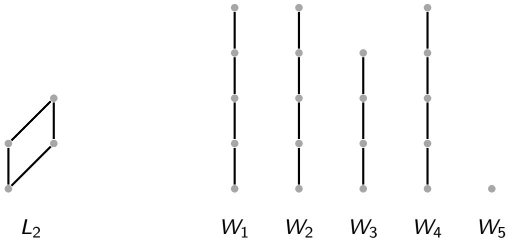

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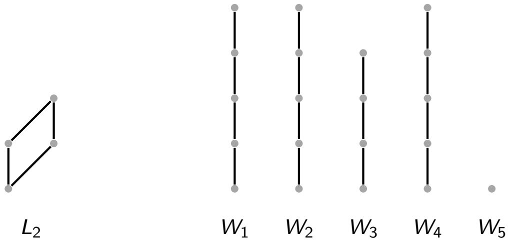

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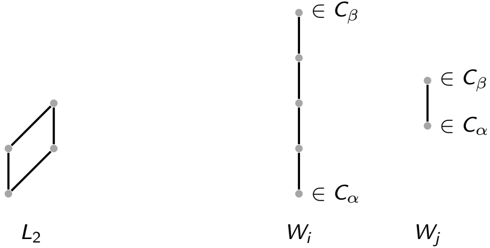

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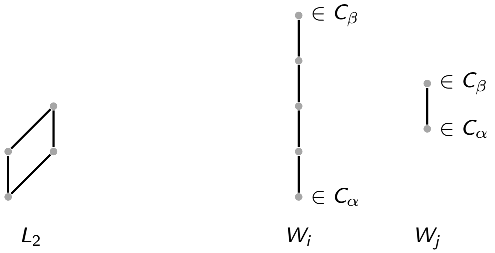

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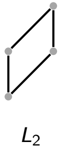

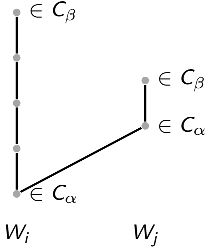

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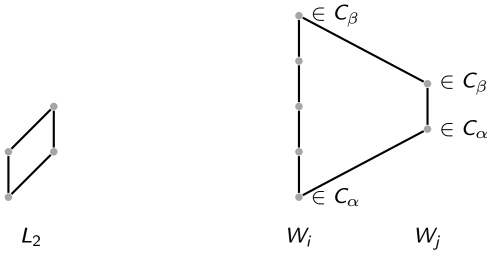

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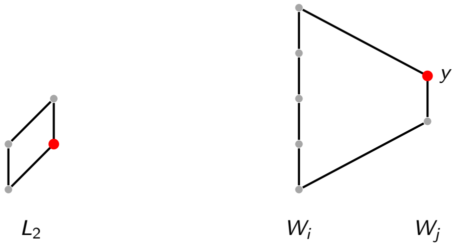

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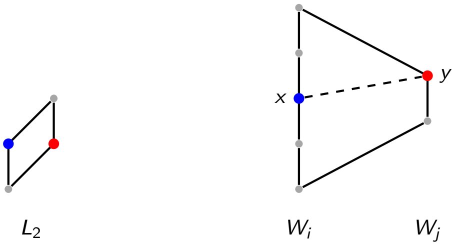

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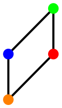

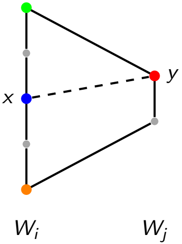

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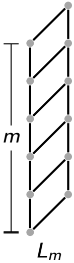
- Similarly $\max W_i > \min W_j$.
- Let $y \in W_j$, and let $x \in W_i$ be incomparable to y.
- Note $\min W_i < x, y < \max W_i$, completing a copy of L_2.

Ladders



- The ladder poset, denoted L_m, has two chains $x_1 < \ldots < x_m$ and $y_1 < \ldots < y_m$ with $x_i \leq y_j$ if and only if $i \leq j$.

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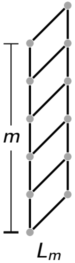

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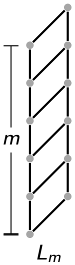

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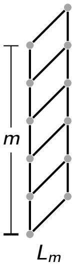
- Bosek–Krawczyk (2015): if $\mathrm{FF}(w, L_m)$ is polynomial in w and m, then there is a polynomial online chain partitioning algorithm.

Ladders



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$$\frac{1}{m-1} w^{\lg(m-1)} \le \mathrm{FF}(w, L_m) \le w^{2.5 \lg 2w + 2 \lg m}$$

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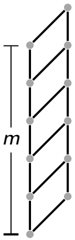

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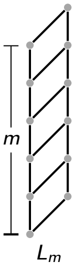

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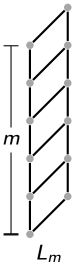
Ladders



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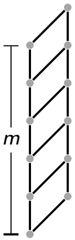

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- The upper bound is superpolynomial but subexponential.

Our results

Definition

- A poset Q is ladder-like if Q can be partitioned into two chains X and Y such that if $x \in X$, $y \in Y$, and x is comparable to y, then $x \leq y$.

Our results

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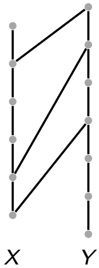

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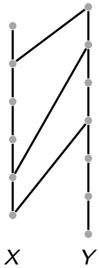

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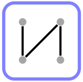

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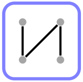

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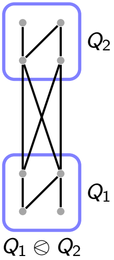

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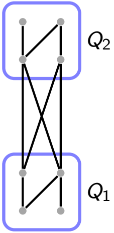

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Our results

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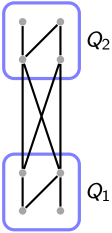

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Our results

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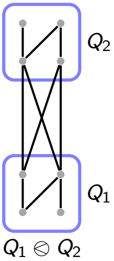

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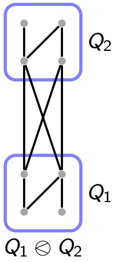

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Theorem (First-Fit Dichotomy)

- *If $Q \in \mathcal{Q}$, then $\mathrm{FF}(w, Q) \leq w^{c_Q \log w}$ for some constant c_Q.*

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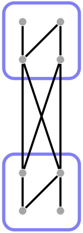

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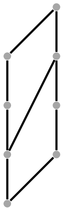
First-Fit Dichotomy Theorem, Upper Bound

Proposition

If Q is a ladder-like m-point poset, then Q is a subposet of L_m.

First-Fit Dichotomy Theorem, Upper Bound



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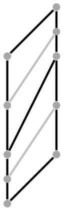

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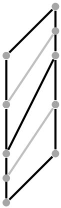

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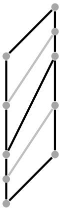

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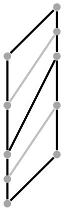

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- Therefore $\mathrm{FF}(w, Q) \leq w^{c_Q \log w}$ when $Q \in \mathcal{Q}$.

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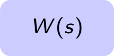

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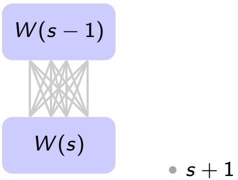

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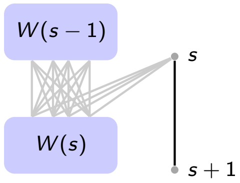

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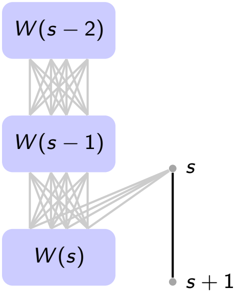

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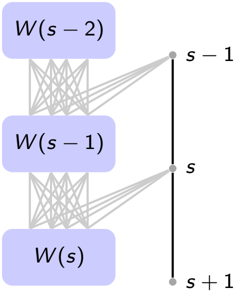

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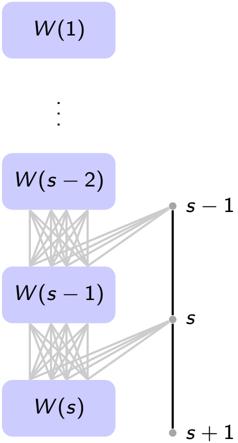

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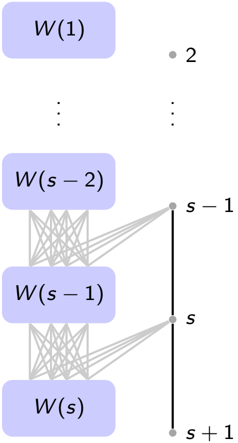

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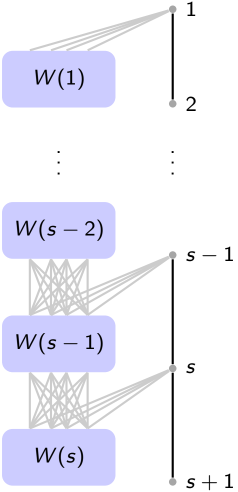

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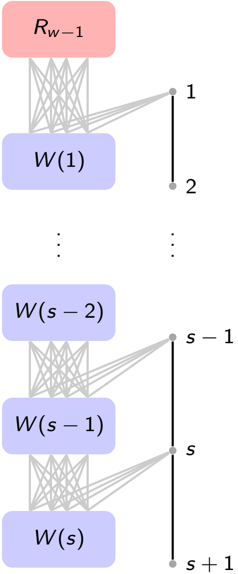

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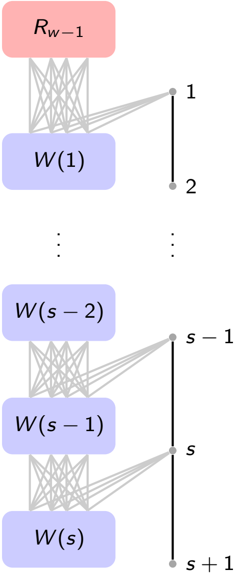

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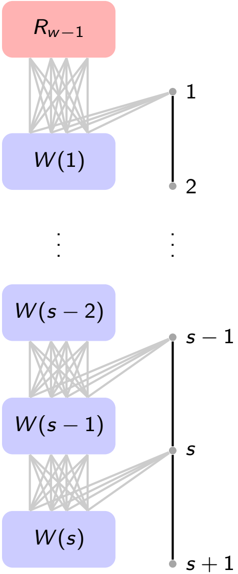

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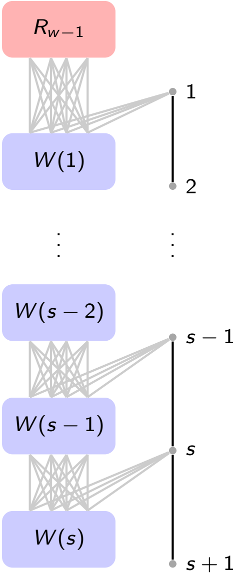

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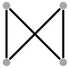
- So, if $Q \notin \mathcal{Q}$, then $\mathrm{FF}(w, Q) \geq 2^w - 1$.

Butterfly Poset



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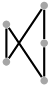

Skewed butterfly $\widehat{B}$

- The butterfly poset B has the form $\overline{2} \olessthan \overline{2}$ where $\overline{2}$ is the 2-element antichain.

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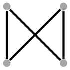

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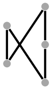

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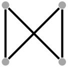

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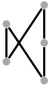

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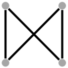

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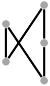

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Theorem

$\mathrm{FF}(w, B) = (1 + o(1))w^{3/2}$

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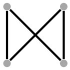

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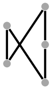

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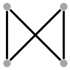
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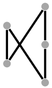
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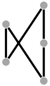
- Both bounds use the Turán number of C_4.
- On the other hand, $\widehat{B} \notin \mathcal{Q}$ and so $\mathrm{FF}(w, \widehat{B}) \geq 2^w - 1$.

Open Problems



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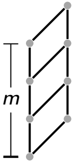
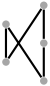


L_m

- Find sharp bounds on $\mathrm{FF}(w, \widehat{B})$.

Open Problems



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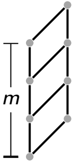
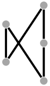


L_m

- Find sharp bounds on $\mathrm{FF}(w, \widehat{B})$.
- For fixed m, is $\mathrm{FF}(w, L_m)$ polynomial in w?

Open Problems



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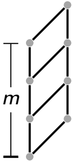

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- Find sharp bounds on $\mathrm{FF}(w, \widehat{B})$.
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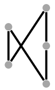
- If yes, then the Dichotomy Theorem becomes stronger:

Open Problems



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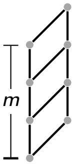

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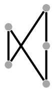
Open Problems



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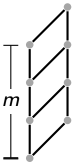

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- Find sharp bounds on $\mathrm{FF}(w, \widehat{B})$.
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- If yes, then the Dichotomy Theorem becomes stronger:
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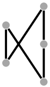
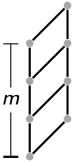
  - Otherwise $\mathrm{FF}(w, Q)$ is at least exponential.

Open Problems



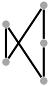
Skewed butterfly $\widehat{B}$



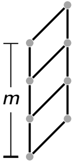
L_m

- Find sharp bounds on $\mathrm{FF}(w, \widehat{B})$.
- For fixed m, is $\mathrm{FF}(w, L_m)$ polynomial in w?
- If yes, then the Dichotomy Theorem becomes stronger:
  - If $Q \in \mathcal{Q}$, then $\mathrm{FF}(w, Q)$ is polynomial.
  - Otherwise $\mathrm{FF}(w, Q)$ is at least exponential.
- If no, then characterize the posets Q such that $\mathrm{FF}(w, Q)$ is polynomial.

Open Problems



Skewed butterfly $\widehat{B}$



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Thank You.