Tight paths in fully directed hypergraphs

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Fully Directed Hypergraphs

- An $r$-graph is an $r$-uniform hypergraph.
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In a fully directed $r$-graph, each edge is a tuple $(u_1, \ldots, u_r)$ of $r$ distinct vertices.
An \( r \)-graph is an \( r \)-uniform hypergraph.

In a fully directed \( r \)-graph, each edge is a tuple \((u_1, \ldots, u_r)\) of \( r \) distinct vertices.

Example: \( V(G) = \{a, b, c, d\} \), \( E(G) = \{(a, b, d), (d, b, c)\} \)
The tight path, denoted by $P_n^{(r)}$, is given by:

$$V(P_n^{(r)}) = \{1, \ldots, n\}$$

$$E(P_n^{(r)}) = \{(i + 1, \ldots, i + r): 0 \leq i \leq n - r\}$$
Paths and Cycles

- The tight path, denoted by $P_n^{(r)}$, is given by:

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  $E(P_n^{(r)}) = \{(i + 1, \ldots, i + r): 0 \leq i \leq n - r\}$

- Example $P_8^{(4)}$: 

\[
\begin{align*}
V(P_8^{(4)}) &= \{1, 2, 3, 4, 5, 6, 7, 0\} \\
E(P_8^{(4)}) &= \{(1, 2, 3, 4), (2, 3, 4, 5), (3, 4, 5, 6), (4, 5, 6, 7), (5, 6, 7, 0), (6, 7, 0, 1), (7, 0, 1, 2)\}
\end{align*}
\]
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Example $P_8^{(4)}$:

1 2 3 4 5 6 7 8

$(1, 2, 3, 4)$
Paths and Cycles

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- Example $P_8^{(4)}$:

  \begin{align*}
  1 & \quad 2 & \quad 3 & \quad 4 & \quad 5 & \quad 6 & \quad 7 & \quad 8 \quad (2, 3, 4, 5)
  \end{align*}
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- Example $P_8^{(4)}$:

  \[
  \begin{array}{cccccc}
  1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
  \end{array}
  \]

  \[(3, 4, 5, 6)\]
Paths and Cycles

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Example $P_8^{(4)}$:

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(4, 5, 6, 7)
Paths and Cycles

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Example $P_8^{(4)}$:

![Diagram of $P_8^{(4)}$ with vertices labeled 1 to 8 and edges indicated by arrows]

$(5, 6, 7, 8)$
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- The tight cycle, denoted by $C_n^{(r)}$, is given by:
  
  $$V(C_n^{(r)}) = \mathbb{Z}_n$$
  
  $$E(C_n^{(r)}) = \{(i + 1, \ldots, i + r): i \in \mathbb{Z}_n\}$$
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  1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
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  \]

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  $V(C_n^{(r)}) = \mathbb{Z}_n$

  $E(C_n^{(r)}) = \{(i + 1, \ldots, i + r) : i \in \mathbb{Z}_n\}$

- Example $C_8^{(4)}$:

  ![Diagram of a tight cycle]

  (0, 1, 2, 3)
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```
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1 & 2 & 3 & 4 & 5 & 6 & 7 & 8
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Example $C_8^{(4)}$:

```
\begin{tikzpicture}
  \node (0) at (0,0) {0};
  \node (1) at (1,0) {1};
  \node (2) at (1,1) {2};
  \node (3) at (0,1) {3};
  \node (4) at (-1,0) {4};
  \node (5) at (-1,-1) {5};
  \node (6) at (0,-1) {6};
  \node (7) at (1,-1) {7};
  \draw[->] (0) -- (1);
  \draw[->] (1) -- (2);
  \draw[->] (2) -- (3);
  \draw[->] (3) -- (4);
  \draw[->] (4) -- (5);
  \draw[->] (5) -- (6);
  \draw[->] (6) -- (7);
  \draw[->] (7) -- (0);
\end{tikzpicture}
```

$(2, 3, 4, 5)$
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**Paths and Cycles**

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- **Example** \( P_8^{(4)} \):

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  \]

- **Example** \( C_8^{(4)} \):

  \[
  \begin{array}{cccccccc}
  3 & 2 & 1 \\
  4 & 0 & \end{array}
  \]

  \[
  (4, 5, 6, 7)
  \]

Paths and Cycles

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- Example $P_{8}^{(4)}$:

  ![Diagram](image)

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1 2 3 4 5 6 7 8
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Example $C_8^{(4)}$:

```
3 2 1 0 4 5 6 7
```

(7, 0, 1, 2)
The extremal function $f(n, r, k)$

- For $0 \leq k \leq r!$, a $k$-orientation of an $r$-graph $H$ is a fully directed $r$-graph $G$ such that for each $e \in E(H)$, exactly $k$ of the $r!$ orderings of the vertices in $e$ are edges in $G$.

- An $(r, k)$-tournament is a $k$-orientation of a complete $r$-graph.

- Note: a $(2, 1)$-tournament is just an ordinary tournament.

- Let $f(n, r, k)$ be the max integer $s$ such that every $n$-vertex $(r, k)$-tournament contains a copy of $P(r)$.
The extremal function $f(n, r, k)$

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- An $(r, k)$-tournament is a $k$-orientation of a complete $r$-graph.
The extremal function $f(n, r, k)$

- For $0 \leq k \leq r!$, a \textit{k-orientation} of an $r$-graph $H$ is a fully directed $r$-graph $G$ such that for each $e \in E(H)$, exactly $k$ of the $r!$ orderings of the vertices in $e$ are edges in $G$.
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The extremal function $f(n, r, k)$

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- Let $f(n, r, k)$ be the max integer $s$ such that every $n$-vertex $(r, k)$-tournament contains a copy of $P_s^{(r)}$. 
The extremal function \( f(n, r, k) \)

- For \( 0 \leq k \leq r! \), a \( k \)-orientation of an \( r \)-graph \( H \) is a fully directed \( r \)-graph \( G \) such that for each \( e \in E(H) \), exactly \( k \) of the \( r! \) orderings of the vertices in \( e \) are edges in \( G \).
- An \( (r, k) \)-tournament is a \( k \)-orientation of a complete \( r \)-graph.
- Note: a \( (2, 1) \)-tournament is just an ordinary tournament.
- Let \( f(n, r, k) \) be the max integer \( s \) such that every \( n \)-vertex \( (r, k) \)-tournament contains a copy of \( P_s^{(r)} \).
- Every tournament has a spanning path: \( f(n, 2, 1) = n \).
Warmup: \( f(n, r, r! - 1) = n \)

- Let \( 2 \leq r \leq n \) and let \( G \) be an \( n \)-vertex \((r, r! - 1)\)-tournament.
Warmup: $f(n, r, r! - 1) = n$

- Let $2 \leq r \leq n$ and let $G$ be an $n$-vertex $(r, r! - 1)$-tournament.
- For each set $S$ of $r$ vertices in $G$, just one ordering of $S$ is absent in $E(G)$. 


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- Let \( u \in V(G) \) and apply induction to obtain a spanning path \( x_1 \cdots x_{n-1} \) in \( G - u \).
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\[ x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6 \quad x_7 \quad x_8 \quad x_9 \quad x_{10} \quad x_{11} \quad x_{12} \quad x_{13} \quad x_{14} \quad x_{15} \]
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Suppose $(u, x_i, \ldots, x_{i+r-2}) \in E(G)$ for some $i$. 
Warmup: $f(n, r, r! - 1) = n$

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- Suppose $(u, x_i, \ldots, x_{i+r-2}) \in E(G)$ for some $i$.
- Let $i$ be the least such integer.
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\begin{itemize}
  \item Suppose $(u, x_i, \ldots, x_{i+r-2}) \in E(G)$ for some $i$.
  \item Let $i$ be the least such integer.
  \item Note $(u, x_{i-1}, \ldots, x_{i+r-3}) \notin E(G)$.
\end{itemize}
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\[ \begin{array}{cccccccccccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
& & & & & & & & & & & & & & & \\
& & & & & & & & & & & & & & & \\
& & & & & & & & & & & & & & & \\
& & & & & & & & & & & & & & & \\
& & & & & & & & & & & & & & & \\
\end{array} \]

- Suppose \((u, x_i, \ldots, x_{i+r-2}) \in E(G)\) for some \( i \).
- Let \( i \) be the least such integer.
- Note \((u, x_{i-1}, \ldots, x_{i+r-3}) \notin E(G)\).
- So \((x_{i-1}, u, x_i, \ldots, x_{i+r-3}) \in E(G)\).
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- Let \( i \) be the least such integer.
- Note \((u, x_{i-1}, \ldots, x_{i+r-3}) \notin E(G)\).
- So \((x_{i-1}, u, x_i, \ldots, x_{i+r-3}) \in E(G)\).
- We may insert \( u \) before \( x_i \).
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- Let $2 \leq r \leq n$ and let $G$ be an $n$-vertex $(r, r! - 1)$-tournament.
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- Let $i$ be the least such integer.
- Note $(u, x_{i-1}, \ldots, x_{i+r-3}) \notin E(G)$.
- So $(x_{i-1}, u, x_i, \ldots, x_{i+r-3}) \in E(G)$.
- We may insert $u$ before $x_i$.
- If no such $i$ exists, then append $u$ at the end.
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![Graph Diagram]

- Suppose $(u, x_i, \ldots, x_{i+r-2}) \in E(G)$ for some $i$.
- Let $i$ be the least such integer.
- Note $(u, x_{i-1}, \ldots, x_{i+r-3}) \not\in E(G)$.
- So $(x_{i-1}, u, x_i, \ldots, x_{i+r-3}) \in E(G)$.
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- We may insert $u$ before $x_i$.
- If no such $i$ exists, then append $u$ at the end.
Let $f(n, r, k)$ be the max integer $s$ such that every $n$-vertex $(r, k)$-tournament contains a copy of $P_s^{(r)}$. 

As $k$ increases from 0 to $r!$, longer paths are forced.

Fix $r$. What is the min. $k$ such that $f(n, r, k)$:

- grows with $n$?
- is polynomial in $n$?
- is linear in $n$?
- equals $n$?

All but the first are open for general $r$. 

Natural Threshold Questions
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- Let $f(n, r, k)$ be the max integer $s$ such that every $n$-vertex $(r, k)$-tournament contains a copy of $P_s^{(r)}$.
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Fix $r$. What is the min. $k$ such that $f(n, r, k)$:

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All but the first are open for general $r$. 
Natural Threshold Questions

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The Pattern Shift Graph

- Tuples \((a_1, \ldots, a_t)\) and \((b_1, \ldots, b_t)\) pattern-match if, for all \(i, j\), we have \(a_i < a_j\) iff \(b_i < b_j\).
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\[
\begin{align*}
132 & \rightarrow 312 \\
231 & \rightarrow 321
\end{align*}
\]
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\[
\text{PSG}_3
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Note: \{132, 231\} is a max. acyclic set in \(\text{PSG}_3\).

\[
\begin{align*}
\text{PSG}_3 \\
123 \\
213 \\
312
\end{align*}
\]
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\[
\begin{array}{c c c}
123 & \text{PSG}_3 & 321 \\
213 & & 231 \\
312 & & 132
\end{array}
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\[
\begin{align*}
123 & \quad \text{PSG}_3 \\
213 & \quad 321 \\
312 & \quad 231 \\
132 & \quad 213
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\[
\begin{array}{c}
123 \\
213 \\
312 \\
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231 \\
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\(\text{PSG}_3\)
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\begin{align*}
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312 & \rightarrow 132, 231 \\
132 & \rightarrow 312, 213 \\
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Growing paths and spanning paths

**Theorem**

Let $k$ and $r$ be constants, and let $a(PSG_r)$ be the maximum size of an acyclic set of vertices in $PSG_r$. We have

$$f(n, r, k) = \begin{cases} O(1) & \text{if } k \leq a(PSG_r) \\ \omega(1) & \text{if } k > a(PSG_r) \end{cases}.$$
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  - Put $(u_1, \ldots, u_r) \in E(G)$ iff $(u_1, \ldots, u_r)$ pattern-matches some permutation in $S$.
- $P(r)$ implies $\text{PSG}_r[A]$ has a walk of size $s - (r - 1)$.
- $A$ is acyclic, so every walk in $\text{PSG}_r[A]$ has size at most $|A|$.
- So $s - (r - 1) \leq |A|$, giving $s \leq |A| + r - 1 = k + r - 1$. 

- If $k > a(\text{PSG}_r)$ and $n \geq R(r)$, then $f(n, r, k) \geq n'$. 

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- If \( k > a(\text{PSG}_r) \) and \( n \geq R^{(r)}(n'; \binom{r}{k}) \), then \( f(n, r, k) \geq n' \).
Growing paths and spanning paths

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For $r \geq 3$, there is a constant $c$ such that

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Let \( k \) and \( r \) be constants, and let \( a(\text{PSG}_r) \) be the maximum size of an acyclic set of vertices in \( \text{PSG}_r \). We have

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▶ Prop: if \( k \geq r!(1 - \frac{1}{e(2r-1)}) \), then \( f(n, r, k) = n \).
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$$\frac{k}{r!}$$
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![Diagram](k/r! a(PSG_r)/r! 1 - 1/e(2r-1) 1)
Growing paths and spanning paths

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Let $k$ and $r$ be constants, and let $a(\text{PSG}_r)$ be the maximum size of an acyclic set of vertices in $\text{PSG}_r$. We have

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Prop: if $k \geq r!(1 - \frac{1}{e(2r-1)})$, then $f(n, r, k) = n$. 

![Diagram showing the relationship between constant paths and spanning paths](image)
Growing paths and spanning paths

Theorem
Let $k$ and $r$ be constants, and let $a(\text{PSG}_r)$ be the maximum size of an acyclic set of vertices in PSG$_r$. We have

$$f(n, r, k) = \begin{cases} O(1) & \text{if } k \leq a(\text{PSG}_r) \\ \omega(1) & \text{if } k > a(\text{PSG}_r) \end{cases}.$$ 

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![Diagram showing constant, growing, and spanning paths](chart.png)
Growing paths and spanning paths

Theorem

Let $k$ and $r$ be constants, and let $a(\text{PSG}_r)$ be the maximum size of an acyclic set of vertices in $\text{PSG}_r$. We have

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\[
\begin{array}{cccc}
\frac{k}{r!} & \text{const. paths} & \frac{c(\ln n)^3}{r} & \frac{a(\text{PSG}_r)}{r!} & \frac{1}{r} & \frac{1}{r!} & 1 - \frac{1}{e(2r-1)} & 1
\end{array}
\]
The case $r = 3$

- Recall: $a(\text{PSG}_3) = 2$. 

---

Warmup:

- $f(n, 3, 5) = f(n, 3, 6) = n$. 

Interesting cases:

- $k = 3$ and $k = 4$. 

Theorem:

$$\Omega(\log n \log \log n) \leq f(n, 3, 3) \leq O(\log n).$$

Theorem:

$$f(n, 3, 4) \geq \Omega(n^{1/5}).$$
The case $r = 3$

- Recall: $a(\text{PSG}_3) = 2$.
- So $f(n, 3, 2) = O(1)$ and $f(n, 3, 3) = \omega(1)$. 

"span. const. polylog $\geq \Omega(n^{1/5})"
The case \( r = 3 \)

- Recall: \( a(\text{PSG}_3) = 2 \).
- So \( f(n, 3, 2) = O(1) \) and \( f(n, 3, 3) = \omega(1) \).
- In fact, we get \( f(n, 3, 2) \leq 3 \) and \( f(n, 3, 3) \geq \Omega(\log \log n) \).
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$$
\Omega \left( \left( \frac{\log n}{\log \log n} \right)^{1/4} \right) \leq f(n, 3, 3) \leq O(\log n).
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$$f(n, 3, 4) \geq \Omega(n^{1/5}).$$
The case \( r = 3 \)

- Recall: \( a(\text{PSG}_3) = 2 \).
- So \( f(n, 3, 2) = O(1) \) and \( f(n, 3, 3) = \omega(1) \).
- In fact, we get \( f(n, 3, 2) \leq 3 \) and \( f(n, 3, 3) \geq \Omega(\log \log n) \).
- Warmup: \( f(n, 3, 5) = f(n, 3, 6) = n \).
- Interesting cases: \( k = 3 \) and \( k = 4 \).

**Theorem**

\[
\Omega \left( \left( \frac{\log n}{\log \log n} \right)^{1/4} \right) \leq f(n, 3, 3) \leq O(\log n).
\]

**Theorem**

\[ f(n, 3, 4) \geq \Omega(n^{1/5}) \]
Outline: \( f(n, 3, 4) \geq \Omega(n^{1/5}) \)

- Let \( G \) be an \( n \)-vertex \((3, 4)\)-tournament.
- Each triple \( \{u, v, w\} \) has 4 orderings in \( E(G) \) and omits 2.
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\[ s \geq \frac{n}{2} \]

\[ t \geq \Omega(n^{1/5}) \]

Let \( T \) be a max. set of vertices in \( G \) not containing a good triple, and let \( t = |T| \).

\[ m \geq \frac{n^3}{t^2} \geq \frac{n^3}{n^2/2} \geq \frac{n^2}{2} \]

\[ s \geq m \geq \frac{n^2}{2} \]

\[ t \geq \Omega(n^{1/5}) \]
Outline: $f(n, 3, 4) \geq \Omega(n^{1/5})$

- Let $G$ be an $n$-vertex $(3, 4)$-tournament.
- Each triple $\{u, v, w\}$ has 4 orderings in $E(G)$ and omits 2.
- Let $s$ be the max. integer such that $P_s^{(3)} \subseteq G$.
- For each $(u, v)$, let $P(uv)$ be a max. path ending $uv$.
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- Say \( (u, v, w) \) is **good** if \( w \in V(P(uv)) \). Say \( \{u, v, w\} \) is **good** if at least one of its orderings is good.
Outline: $f(n, 3, 4) \geq \Omega(n^{1/5})$

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- Let $m$ be num. of good triples in $G$. Note $s \geq \frac{m}{n^2}$.
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Outline: $f(n, 3, 4) \geq \Omega(n^{1/5})$

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- deCaen: $m \geq \binom{n}{3} \geq \binom{n-2}{3} \geq \frac{(n-2)^3}{3t^2}$.
- $sn^2 \geq m \geq \frac{(n-2)^3}{3t^2}$ and so $t \geq \Omega((n/s)^{1/2})$. 
Outline: $f(n, 3, 4) \geq \Omega(n^{1/5})$
Outline: \( f(n, 3, 4) \geq \Omega(n^{1/5}) \)

\[ |T| \geq \Omega((n/s)^{1/2}) \text{ and } T \text{ has no good triple.} \]
Outline: $f(n, 3, 4) \geq \Omega(n^{1/5})$

- $|T| \geq \Omega((n/s)^{1/2})$ and $T$ has no good triple.
- Suppose $uvw \in E(G[T])$. 

![Diagram of a graph $G$ with a subgraph $T$ containing vertices $u$, $v$, and $w$. The edge $uvw$ is included in $E(G[T])$.](image-url)
Outline: \( f(n, 3, 4) \geq \Omega(n^{1/5}) \)

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- So \( s_{vw} > s_{uv} \). Thus \( G[T] \) is acyclic.
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- Lemma: The maximum paths in a $(3, 4)$-tournament with no $C_3^{(3)}$ are pairwise intersecting.
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- This implies $G[T]$ contains a tight path on $\left\lceil \sqrt{|T|} \right\rceil$ vertices.
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- Lemma: The maximum paths in a \((3, 4)\)-tournament with no \( C_3^{(3)} \) are pairwise intersecting.
- This implies \( G[T] \) contains a tight path on \( \lceil \sqrt{|T|} \rceil \) vertices.
- So \( G \) has a path on at least \( \max\{s, \Omega((n/s)^{1/4})\} \) vertices.
Growing paths in general fully directed $r$-graphs

Let $n(r) = n(n-1) \cdots (n-(r-1)) = (1 - o(1))n^r$. 

So $(1 - 1/r)$ is the density threshold for growing paths in fully directed $r$-graphs.

With $k! = 1 - 1/r - 1/r!$, $(r, k)$-tournaments have growing paths.

The even distribution requirement of tournaments forces growing paths at lower densities.
Growing paths in general fully directed $r$-graphs

Let $n(r) = n(n - 1) \cdots (n - (r - 1)) = (1 - o(1)) n^r$.

Theorem
For each $r$ and each positive $\varepsilon$, for sufficiently large $n$, there is an $n$-vertex fully directed $r$-graph $G$ with $|E(G)| \geq (1 - \frac{1}{r} - \varepsilon) n(r)$ such that every path in $G$ has size at most $r^3 / \varepsilon$. 
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**Theorem**

For each $r$ and each $s$, for all sufficiently large $n$, every $n$-vertex fully directed $r$-graph $G$ with $|E(G)| \geq (1 - \frac{1}{n}) n_{(r)}$ contains a path of size $s$. 
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Open Problems

- Improve the bounds $\Omega(n^{1/5}) \leq f(n, 3, 4) \leq n$. 

- Does every $(3, 4)$-tournament have a spanning path?

- Improve the bounds $(1 - c (\ln r)^{3/r}) r! \leq k = a(PSG_r) \leq (1 - 1/r - 2/r!) r!$ on the threshold $k$ for growing paths in $(r, k)$-tournaments.

- There are polynomial paths in fully directed 3-graphs at the density threshold $2/3$.

- For $r \geq 4$, do fully directed $r$-graphs at the growing paths density threshold $1 - 1/r$ also have polynomial paths?

- What is the threshold on $k$ for $(r, k)$-tournaments to have polynomial paths?
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Open Problems

- Improve the bounds $\Omega(n^{1/5}) \leq f(n, 3, 4) \leq n$.
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- Improve the bounds $\left(1 - \frac{c(\ln r)^3}{r}\right)r! \leq k = a(PSG_r) \leq \left(1 - \frac{1}{r} - \frac{2}{r!}\right)r!$ on the threshold $k$ for growing paths in $(r, k)$-tournaments.
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Open Problems

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- What is the threshold on $k$ for $(r, k)$-tournaments to have polynomial paths? spanning paths?

Thank You.