Online coloring blowups of a known graph

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A motivating problem

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The greedy algorithm uses at most \( 2w - 1 \) colors.
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\text{k} \\
\text{1}
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There is a strategy for Spoiler that forces Algorithm to use at least \( \left\lceil \frac{3}{2} w \right\rceil \) colors.

- This forces \( 3k \) colors in a game of width \( 2k \).
A generalization to graphs

- For a graph $G$, the $G$-coloring game of width $w$ is played in rounds between Spoiler and Algorithm:

  ▶ Spoiler chooses a vertex $v \in V(G)$ and plays a token $x$ at $v$.
  ▶ Algorithm assigns $x$ a color.
  ▶ The associated token graph $H$ is obtained from $G$ by replacing each vertex $v$ with the complete graph on the tokens at $v$.
  ▶ Algorithm must give a proper coloring of $H$ and wants to minimize the number of colors used.
  ▶ Spoiler must ensure that $\chi(H) \leq w$ and wants to force many colors.
  ▶ The value of the game, denoted $f(G;w)$, is the number of colors needed by an optimal strategy for Algorithm.
  ▶ Let $G$ be the graph on $\mathbb{R}$ with $uv \in E(G)$ if and only if $|u-v| < 1$.
  ▶ $\left\lfloor \frac{3}{2}w \right\rfloor \leq f(G;w) \leq 2w - 1$. 
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- $\left\lfloor \frac{3}{2}w \right\rfloor \leq f(G; w) \leq 2w - 1$. 
A graph $G$ is **online-perfect** if $f(G; w) = w$. 
Bipartite graphs

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Every bipartite graph is online-perfect.
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- Choose a linear ordering on a set of $w$ colors.
- Tokens played at the left part are assigned colors greedily in order.
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- A conflict would imply the token graph has a clique on more than $w$ vertices.
Twins

Vertices \( u \) and \( u' \) are **twins** in \( G \) if they have the same neighborhood in \( G - \{u, u'\} \).
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- Given $u \in V(G)$, we may clone $u$ to produce a new graph $G'$ with an additional vertex $u'$ that is a twin of $G$. 
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![Diagram of twin vertices with shared neighborhood]
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- Both $uu' \in E(G)$ and $uu' \notin E(G)$ are possible.
- Given $u \in V(G)$, we may **clone** $u$ to produce a new graph $G'$ with an additional vertex $u'$ that is a twin of $G$.
- Fact: a graph $G$ is $P_4$-free if and only if $G$ is obtainable from a single vertex by cloning.
Proposition

If $G'$ is obtained from $G$ by cloning $u$, then $f(G'; w) = f(G; w)$. 
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- $f(G; w) \leq f(G'; w)$: clear since $G'$ has an induced copy of $G$. 

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**Corollary**

If $G$ is obtainable from a bipartite graph by cloning, then $G$ is online-perfect.
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\[ f(G; w) \leq f(G'; w) \]: clear since \( G' \) has an induced copy of \( G \).

\[ f(G'; w) \leq f(G; w) \]: adapt an optimal strategy for \( G \).
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Let $G$ be a graph. The following are equivalent.

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3. $G$ does not have an induced copy of any of the following:
   - $C_n$ for odd $n$ at least 5
   - $C_5^+$
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- (1) $\rightarrow$ (2): clear
- (2) $\rightarrow$ (3): Spoiler Lemma
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- (2) $\rightarrow$ (3): Spoiler Lemma
- (3) $\rightarrow$ (4): roughly a page of structural graph theory.
Theorem

Let $G$ be a graph. The following are equivalent.

1. $G$ is online-perfect.
2. $f(G; 2) = 2$.
3. $G$ does not have an induced copy of any of the following:
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   - $C_5^+$
   - $P_5^2$
   - The bull
4. $G$ is obtainable from a bipartite graph by cloning vertices.

- $(1) \rightarrow (2)$: clear
- $(2) \rightarrow (3)$: Spoiler Lemma
- $(3) \rightarrow (4)$: roughly a page of structural graph theory.
- $(4) \rightarrow (1)$: previous corollary
Characterization of online-perfect graphs

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Cor: $P_4$-free graphs $\subset$ online-perfect graphs $\subset$ perfect graphs
Characterization of online-perfect graphs

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4. $G$ is obtainable from a bipartite graph by cloning vertices.

- Cor: $P_4$-free graphs $⊊$ online-perfect graphs $⊊$ perfect graphs
- 4’: $G$ is online-perfect if and only if it the result of replacing each vertex in a bipartite graph with a $P_4$-free graph.
A Spoiler strategy

Lemma (Spoiler Lemma)

Let $U \subseteq V(G)$, where $U = \{u_1, \ldots, u_t\}$, and suppose that:

1. There are vertices $x$ and $y$ such that $u_1xyu_t$ is a path and $G[U \cup \{x, y\}]$ is bipartite, and
2. For each $i$, there is a common neighbor $z_i$ of $u_i$ and $u_i+1$ such that $G[U \cup \{z_i\}]$ is bipartite.

If $w$ is even, then $f(G; w) \geq (1 + \frac{1}{2})w$. 

\[ u_1 \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \ u_t \]
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![Diagram]

$u_1 \quad \cdots \quad \cdots \quad u_t$

$z_i$
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Corollary

If $n$ is odd and $n \geq 5$ and $w$ is even, then $f(C_n; w) \geq \frac{n}{n-1}w$. 

$C_n$
Minimal non-online-perfect graphs: lower bounds

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- Apply Spoiler Lemma with $t = (n-1)/2$. 
Lemma (Spoiler Lemma)

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If $w$ is even, then $f(G; w) \geq (1 + \frac{1}{2t})w$.

Corollary

If $G \in \{C^+_5, P^2_5\}$ and $w$ is even, then $f(G; w) \geq \frac{5}{4}w$. 
Lemma (Spoiler Lemma)

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Corollary

If $G \in \{C_5^+, P_5^2\}$ and $w$ is even, then $f(G; w) \geq \frac{5}{4}w$.

Apply Spoiler Lemma with $t = 2$. 
Lemma (Spoiler Lemma)

Let $U \subseteq V(G)$, where $U = \{u_1, \ldots, u_t\}$, and suppose that:

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If $w$ is even, then $f(G; w) \geq (1 + \frac{1}{2t})w$.

Corollary

If $G$ is the bull graph and $w$ is even, then $f(G; w) \geq \frac{7}{6}w$. 
Minimal non-online-perfect graphs: lower bounds

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If $w$ is even, then $f(G; w) \geq (1 + \frac{1}{2t})w$.

**Corollary**

If $G$ is the bull graph and $w$ is even, then $f(G; w) \geq \frac{7}{6}w$.

- Apply Spoiler Lemma with $t = 3$. 

![Graph Diagram]
Minimal non-online-perfect graphs: upper bounds

Proposition (Fractional Coloring Strategy)

If $G$ has a $(p, q)$-coloring, then $f(G; w) \leq p \left\lceil \frac{w}{2q} \right\rceil$. In particular, $f(G; w) \leq \left( \frac{1}{2} \chi_f(G) + o(1) \right) w$. 
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If \( n \) is odd, then \( f(C_n; w) \leq \left( \frac{n}{n-1} + o(1) \right) w \).
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Corollary
If $n$ is odd, then $f(C_n; w) \leq \left( \frac{n}{n-1} + o(1) \right)w$.

- Apply Prop. with $\chi_f(C_n) = \frac{2n}{n-1}$. 

$C_n$
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If $G \in \{C_5^+, P_5^2\}$, then $f(G; w) \leq \left(\frac{3}{2} + o(1)\right)w$. 
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If $G \in \{C_5^+, P_5^2\}$, then $f(G; w) \leq (\frac{3}{2} + o(1))w$.

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If $G$ is the bull graph and $w$ is even, then $f(G; w) \leq \left( \frac{3}{2} + o(1) \right)w$.

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Summary

Theorem

A graph is online-perfect if and only if it does not have an induced copy of any of the following:

- $C_n$ for odd $n$ at least 5
- $C_5^+$
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- The bull

Open Problems

Determine the asymptotics of $f(C_n^+; w)$, $f(P_5^2; w)$, $f(G; w)$.

Thank You.
Summary

Theorem

A graph is online-perfect if and only if it does not have an induced copy of any of the following:

\[ C_n \text{ for odd } n \geq 5 \quad \quad C_5^+ \quad \quad P_5^2 \quad \quad \text{The bull} \]

- For odd \( n \) and \( n \geq 5 \), we have \( f(C_n; w) = \left( \frac{n}{n-1} + o(1) \right)w \).
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Theorem

A graph is online-perfect if and only if it does not have an induced copy of any of the following:

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For odd $n$ and $n \geq 5$, we have $f(C_n; w) = \left( \frac{n}{n-1} + o(1) \right)w$.

$(\frac{5}{4} - o(1))w \leq f(C_5^+; w) \leq (\frac{3}{2} + o(1))w.$
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Thm: $f(P_5^2; w) = (\frac{5}{4} + o(1))w$.
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- Thm: $f(P_5^2; w) = (\frac{5}{4} + o(1))w$.
- For the bull $B$: $(\frac{7}{6} - o(1))w \leq f(B; w) \leq (\frac{3}{2} + o(1))w$. 
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Open Problems

Determine the asymptotics of $f(C_5^+; w)$,
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Determine the asymptotics of $f(C_5^+; w)$, $f(B; w)$,
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Theorem

A graph is online-perfect if and only if it does not have an induced copy of any of the following:

- $C_n$ for odd $n$ at least 5
- $C^+_5$
- $P^2_5$
- The bull

- For odd $n$ and $n \geq 5$, we have $f(C_n; w) = \left(\frac{n}{n+1} + o(1)\right)w$.
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- Thm: $f(P^2_5; w) = (\frac{5}{4} + o(1))w$.
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Determine the asymptotics of $f(C^+_5; w)$, $f(B; w)$, $f(G; w)$.
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