Turán and Ramsey Results for Boolean Algebras

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Boolean Algebras

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- Given disjoint sets $X_0, X_1, \ldots, X_d$, with $X_i \neq \emptyset$ for $i \geq 1$, the generated $d$-dimensional Boolean algebra is the family of all sets formed by the union of $X_0$ with 0 or more members of $\{X_1, \ldots, X_d\}$. 

Such a family of $2^d$ sets forms a copy of $B^d$.

A family is $B^d$-free if it does not contain a copy of $B^d$. 


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Such a family of $2^d$ sets forms a copy of $\mathcal{B}_d$.

A family is $\mathcal{B}_d$-free if it does not contain a copy of $\mathcal{B}_d$. 
Turán Problem

What is the largest size of a $\mathcal{B}_d$-free subfamily of $2^{[n]}$?

Prior Work

Theorem $b(n, d) \leq 50 \cdot n - \frac{1}{2}d \cdot 2^n$. 
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- What is the largest size of a $B_d$-free subfamily of $2^{[n]}$?
- Let $b(n, d) = \max \{|F| : F \subseteq 2^{[n]} \text{ and } F \text{ is } B_d \text{-free}\}$. 

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- [Erdős–Kleitman 1971] For some constants $c_1, c_2$ and $n$ sufficiently large

$$c_1 \cdot n^{-1/4} \cdot 2^n \leq b(n, 2) \leq c_2 \cdot n^{-1/4} \cdot 2^n.$$
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[Gunderson–Rödl–Sidorenko 1999] For each $d$, there exists $c_d$ such that for $n$ sufficiently large

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- For $d \geq 1$, the bounds $(2n)^{1 - \frac{2}{2^d}} \leq \alpha_d(n) \leq (4n)^{1 - \frac{2}{2^d}}$ hold.
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- For fixed $d$, we have $\alpha_d(n) = (1 + o(1))(2n)^{1 - \frac{2}{2^d}}$.
- For $d \geq 1$, we have $\left(\frac{\alpha_d(n)}{2}\right)/n = \alpha_{d-1}(n)$.
Szemerédi’s Cube Lemma

- Given $x_0, x_1, \ldots, x_d$ with $x_0 \geq 0$ and $x_i \geq 1$ for $i \geq 1$, the generated affine $d$-cube is the set of all integers obtained by adding $x_0$ to the sum of 0 or more members of \{x_1, \ldots, x_d\}.
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If $A \subseteq [0, n]$ and $|A| > \alpha_d(n)$, then $A$ contains an affine $d$-cube.
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- Using $\alpha_d(n) \leq (4n)^{1-\frac{2}{2d}} < 4n^{1-\frac{2}{2d}}$, we obtain:

Corollary
If $A \subseteq [0, n]$ and $|A| \geq 4n^{1-\frac{2}{2d}}$, then $A$ contains an affine $d$-cube.
The Lubell Function

Given $\mathcal{F} \subseteq 2^{[n]}$, let $X$ be the number of times a random full chain meets $\mathcal{F}$. The Lubell function of $\mathcal{F}$, denoted $h_n(\mathcal{F})$, is $E[X]$. Think of $h_n(\mathcal{F})$ as a measure of the size of $\mathcal{F}$, with $0 \leq h_n(\mathcal{F}) \leq n+1$. 
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$\mathcal{F} \subseteq 2^{[n]}$ means $\mathcal{F}$ is a subset of the powerset of $\{1, 2, \ldots, n\}$. A full chain is a sequence of elements from $[n]$ that are all in $\mathcal{F}$. The expectation $E[X]$ calculates the average number of full chains that meet $\mathcal{F}$. The Lubell function $h_n(\mathcal{F})$ quantifies this measure.
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- $J_2$ also gives useful information.
- For each ordered pair $(A, B)$ of distinct elements in $\mathcal{F}$ with $A \subsetneq B$, let $Y_{A,B}$ be the indicator r.v. for the full chain containing $A$ and $B$. 

\[ E[J_2] = \sum_{A,B} E[Y_{A,B}] = \sum_{A,B} 1^{(n|A|, |B|−|A|, n−|B|)} \]

where $\mathcal{F}_S$ is the set of all $A \in \mathcal{F}$ that are disjoint from $S$ with $A \cup S \in \mathcal{F}$. 

\[ \begin{array}{c}
\text{[n]} \\
\text{∅}
\end{array} \] 

\[ B \]

\[ A \]
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where \(\mathcal{F}_S\) is the set of all \(A \in \mathcal{F}\) that are disjoint from \(S\) with \(A \cup S \in \mathcal{F}\).
Extension of Szemerédi’s Cube Lemma

Theorem
If $\mathcal{F} \subseteq 2^{[n]}$ and $h_n(\mathcal{F}) > \alpha_d(n)$, then $\mathcal{F}$ contains a copy of $B_d$. 
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**Theorem**
If \( F \subseteq 2^{[n]} \) and \( h_n(F) > \alpha_d(n) \), then \( F \) contains a copy of \( B_d \).

**Corollary (Szemerédi’s Cube Lemma)**
If \( A \subseteq [0, n] \) and \( |A| > \alpha_d(n) \), then \( A \) contains an affine \( d \)-cube.
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Corollary (Szemerédi’s Cube Lemma)
If $A \subseteq [0, n]$ and $|A| > \alpha_d(n)$, then $A$ contains an affine $d$-cube.

Proof.

Let $F = \bigcup_{k \in A} \binom{[n]}{k}$.
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If \( \mathcal{F} \subseteq 2^{[n]} \) and \( h_n(\mathcal{F}) > \alpha_d(n) \), then \( \mathcal{F} \) contains a copy of \( B_d \).

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If \( A \subseteq [0, n] \) and \( |A| > \alpha_d(n) \), then \( A \) contains an affine \( d \)-cube.

Proof.

\( \blacktriangleright \) Let \( \mathcal{F} = \bigcup_{k \in A} ([n]) \).

\( \blacktriangleright \) Note \( \mathcal{F} \subseteq 2^{[n]} \) and \( h_n(\mathcal{F}) = |A| > \alpha_d(n) \).
Extension of Szemerédi’s Cube Lemma

Theorem
If \( \mathcal{F} \subseteq 2^{[n]} \) and \( h_n(\mathcal{F}) > \alpha_d(n) \), then \( \mathcal{F} \) contains a copy of \( B_d \).

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- Let \( \mathcal{F} = \bigcup_{k \in A} ([n]) \).
- Note \( \mathcal{F} \subseteq 2^{[n]} \) and \( h_n(\mathcal{F}) = |A| > \alpha_d(n) \).
- By the theorem: \( \mathcal{F} \) contains a copy of \( B_d \) generated by disjoint sets \( X_0, X_1, \ldots, X_d \).
Extension of Szemerédi’s Cube Lemma

Theorem
If $\mathcal{F} \subseteq 2^{[n]}$ and $h_n(\mathcal{F}) > \alpha_d(n)$, then $\mathcal{F}$ contains a copy of $B_d$.

Corollary (Szemerédi’s Cube Lemma)
If $A \subseteq [0, n]$ and $|A| > \alpha_d(n)$, then $A$ contains an affine $d$-cube.

Proof.

1. Let $\mathcal{F} = \bigcup_{k \in A} ([n])_k$.
2. Note $\mathcal{F} \subseteq 2^{[n]}$ and $h_n(\mathcal{F}) = |A| > \alpha_d(n)$.
3. By the theorem: $\mathcal{F}$ contains a copy of $B_d$ generated by disjoint sets $X_0, X_1, \ldots, X_d$.
4. Hence $A$ contains an affine $d$-cube generated by $x_0, \ldots, x_d$ with $x_i = |X_i|$.
Extension of Szemerédi’s Cube Lemma

Theorem
If $\mathcal{F} \subseteq 2^{[n]}$ and $h_n(\mathcal{F}) > \alpha_d(n)$, then $\mathcal{F}$ contains a copy of $\mathcal{B}_d$.

Corollary (Szemerédi’s Cube Lemma)
If $A \subseteq [0, n]$ and $|A| > \alpha_d(n)$, then $A$ contains an affine $d$-cube.

Question

- Is it true that among all $\mathcal{B}_d$-free families $\mathcal{F} \subseteq 2^{[n]}$ that maximize $h_n(\mathcal{F})$, at least one is the union of level sets?
Extension of Szemerédi’s Cube Lemma

**Theorem**
If $\mathcal{F} \subseteq 2^{[n]}$ and $h_n(\mathcal{F}) > \alpha_d(n)$, then $\mathcal{F}$ contains a copy of $B_d$.

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If $A \subseteq [0, n]$ and $|A| > \alpha_d(n)$, then $A$ contains an affine $d$-cube.

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- Is it true that among all $B_d$-free families $\mathcal{F} \subseteq 2^{[n]}$ that maximize $h_n(\mathcal{F})$, at least one is the union of level sets?
- If so, then both extremal problems are equivalent.
Extension of Szemerédi’s Cube Lemma

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If $\mathcal{F} \subseteq 2^{[n]}$ and $h_n(\mathcal{F}) > \alpha_d(n)$, then $\mathcal{F}$ contains a copy of $B_d$.

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If $A \subseteq [0, n]$ and $|A| > \alpha_d(n)$, then $A$ contains an affine $d$-cube.

Question

- Is it true that among all $B_d$-free families $\mathcal{F} \subseteq 2^{[n]}$ that maximize $h_n(\mathcal{F})$, at least one is the union of level sets?
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- Sperner’s Theorem: yes for $d = 1$. 
Extension of Szemerédi’s Cube Lemma

Theorem
If \( \mathcal{F} \subseteq 2^{[n]} \) and \( h_n(\mathcal{F}) > \alpha_d(n) \), then \( \mathcal{F} \) contains a copy of \( \mathcal{B}_d \).

Corollary (Szemerédi’s Cube Lemma)
If \( A \subseteq [0, n] \) and \( |A| > \alpha_d(n) \), then \( A \) contains an affine \( d \)-cube.

Question

- Is it true that among all \( \mathcal{B}_d \)-free families \( \mathcal{F} \subseteq 2^{[n]} \) that maximize \( h_n(\mathcal{F}) \), at least one is the union of level sets?
- If so, then both extremal problems are equivalent.
- Sperner’s Theorem: yes for \( d = 1 \).
- Open for \( d \geq 2 \).
Extension of Szemerédi’s Cube Lemma: Proof

Theorem
If $\mathcal{F} \subseteq 2^{[n]}$ and $h_n(\mathcal{F}) > \alpha_d(n)$, then $\mathcal{F}$ contains a copy of $B_d$. 
Extension of Szemerédi’s Cube Lemma: Proof

Theorem
If $\mathcal{F} \subseteq 2^{[n]}$ and $h_n(\mathcal{F}) > \alpha_d(n)$, then $\mathcal{F}$ contains a copy of $B_d$.

- By induction on $d$. Case $d = 0$: trivial.
Extension of Szemerédi’s Cube Lemma: Proof

**Theorem**

If $\mathcal{F} \subseteq 2^{[n]}$ and $h_n(\mathcal{F}) > \alpha_d(n)$, then $\mathcal{F}$ contains a copy of $B_d$.

- By induction on $d$. Case $d = 0$: trivial.
- Let $X$ be the number of times a random full chain meets $\mathcal{F}$. 
Theorem
If $\mathcal{F} \subseteq 2^{[n]}$ and $h_n(\mathcal{F}) > \alpha_d(n)$, then $\mathcal{F}$ contains a copy of $\mathcal{B}_d$.

- By induction on $d$. Case $d = 0$: trivial.
- Let $X$ be the number of times a random full chain meets $\mathcal{F}$.
- $E[X] = h_n(\mathcal{F}) > \alpha_d(n)$. 

\[\emptyset \subseteq [n]\]
Extension of Szemerédi’s Cube Lemma: Proof

Theorem
If $\mathcal{F} \subseteq 2^{[n]}$ and $h_n(\mathcal{F}) > \alpha_d(n)$, then $\mathcal{F}$ contains a copy of $B_d$.

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- Let $X$ be the number of times a random full chain meets $\mathcal{F}$.
- $E[X] = h_n(\mathcal{F}) > \alpha_d(n)$.
- By convexity:
  $$E\left[\binom{X}{2}\right] \geq \binom{E[X]}{2}.$$
Extension of Szemerédi’s Cube Lemma: Proof

**Theorem**

If $\mathcal{F} \subseteq 2^{[n]}$ and $h_n(\mathcal{F}) > \alpha_d(n)$, then $\mathcal{F}$ contains a copy of $\mathcal{B}_d$.

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  \[
  E\left[\binom{X}{2}\right] \geq \binom{E[X]}{2} > \binom{\alpha_d(n)}{2}
  \]
Theorem
If $\mathcal{F} \subseteq 2^{[n]}$ and $h_n(\mathcal{F}) > \alpha_d(n)$, then $\mathcal{F}$ contains a copy of $\mathcal{B}_d$.

- By induction on $d$. Case $d = 0$: trivial.
- Let $X$ be the number of times a random full chain meets $\mathcal{F}$.
- $\mathbb{E}[X] = h_n(\mathcal{F}) > \alpha_d(n)$.
- By convexity:
  $$\mathbb{E}[(\binom{X}{2})] \geq \binom{\mathbb{E}[X]}{2} > \binom{\alpha_d(n)}{2} = n\alpha_{d-1}(n)$$
Theorem

If $\mathcal{F} \subseteq 2^{[n]}$ and $h_n(\mathcal{F}) > \alpha_d(n)$, then $\mathcal{F}$ contains a copy of $B_d$.

By convexity:

$$E[(\binom{X}{2})] \geq (E[X]^2) > \binom{\alpha_d(n)}{2} = n\alpha_{d-1}(n)$$
Extension of Szemerédi’s Cube Lemma: Proof

Theorem

If $\mathcal{F} \subseteq 2^{[n]}$ and $h_n(\mathcal{F}) > \alpha_d(n)$, then $\mathcal{F}$ contains a copy of $B_d$.

- By convexity:
  $$\mathbb{E}[(X)^2] \geq \left(\frac{\mathbb{E}[X]}{2}\right) > \left(\frac{\alpha_d(n)}{2}\right) = n\alpha_{d-1}(n)$$

- Grouping pairs $(A, B) \in \mathcal{F} \times \mathcal{F}$ with $A \subsetneq B$ by $B - A$, with $S = B - A$:

  $$\mathbb{E}[(X)^2] =$$
Extension of Szemerédi’s Cube Lemma: Proof

Theorem
If \( \mathcal{F} \subseteq 2^{[n]} \) and \( h_n(\mathcal{F}) > \alpha_d(n) \), then \( \mathcal{F} \) contains a copy of \( B_d \).

- By convexity:
  \[
  \mathbb{E}[\binom{X}{2}] \geq \left( \frac{\mathbb{E}[X]}{2} \right) > \left( \frac{\alpha_d(n)}{2} \right) = n \alpha_{d-1}(n)
  \]

- Grouping pairs \((A, B) \in \mathcal{F} \times \mathcal{F}\) with \( A \subsetneq B \) by \( B - A \), with \( S = B - A \):
  \[
  \mathbb{E}[\binom{X}{2}] = \sum_{k=1}^{n} \frac{1}{\binom{n}{k}} \sum_{S \in \binom{[n]}{k}} h_{n-k}(\mathcal{F}_S),
  \]
Theorem

If $\mathcal{F} \subseteq 2^{[n]}$ and $h_n(\mathcal{F}) > \alpha_d(n)$, then $\mathcal{F}$ contains a copy of $B_d$.

- By convexity:
  $$E[(\chi_2)] \geq (\frac{E[X]}{2}) > (\frac{\alpha_d(n)}{2}) = n\alpha_{d-1}(n)$$

- Grouping pairs $(A, B) \in \mathcal{F} \times \mathcal{F}$ with $A \subset B$ by $B - A$, with $S = B - A$:
  $$E[(\chi_2)] = \sum_{k=1}^{n} \frac{1}{\binom{n}{k}} \sum_{S \in \binom{[n]}{k}} h_{n-k}(\mathcal{F}_S),$$
  where $\mathcal{F}_S$ is the family of all $A \in \mathcal{F}$ that are disjoint from $S$ with $A \cup S \in \mathcal{F}$.
Extension of Szemerédi’s Cube Lemma: Proof

Theorem
If $F \subseteq 2^{[n]}$ and $h_n(F) > \alpha_d(n)$, then $F$ contains a copy of $B_d$.

- By convexity:
  \[ E[(X_2)^n] \geq E[X_2] > \alpha_d(n) = n\alpha_{d-1}(n) \]

- Grouping pairs $(A, B) \in F \times F$ with $A \subsetneq B$ by $B - A$, with $S = B - A$:
  \[ E[(X_2)^n] = \sum_{k=1}^{n} \frac{1}{(n\choose k)} \sum_{S \in \binom{[n]}{k}} h_{n-k}(F_S), \]
  where $F_S$ is the family of all $A \in F$ that are disjoint from $S$ with $A \cup S \in F$.

  \[ \sum_{k=1}^{n} \frac{1}{(n\choose k)} \sum_{S \in \binom{[n]}{k}} h_{n-k}(F_S) > n\alpha_{d-1}(n) \]
Extension of Szemerédi’s Cube Lemma: Proof

**Theorem**

If $\mathcal{F} \subseteq 2^{[n]}$ and $h_n(\mathcal{F}) > \alpha_d(n)$, then $\mathcal{F}$ contains a copy of $B_d$.

\[ \sum_{k=1}^{n} \frac{1}{\binom{n}{k}} \sum_{S \in \binom{[n]}{k}} h_{n-k}(\mathcal{F}_S) > n\alpha_{d-1}(n) \]
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If \( \mathcal{F} \subseteq 2^{[n]} \) and \( h_n(\mathcal{F}) > \alpha_d(n) \), then \( \mathcal{F} \) contains a copy of \( B_d \).

\[
\frac{1}{\binom{n}{k}} \sum_{S \in \binom{[n]}{k}} h_{n-k}(\mathcal{F}_S) > n\alpha_{d-1}(n)
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\frac{1}{\binom{n}{k}} \sum_{S \in \binom{[n]}{k}} h_{n-k}(\mathcal{F}_S) > \alpha_{d-1}(n)
\]

Find \( k \) such that

Find \( S \in \binom{[n]}{k} \) with \( h_{n-k}(\mathcal{F}_S) > \alpha_{d-1}(n) \).

\( \emptyset \)
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If $\mathcal{F} \subseteq 2^{[n]}$ and $h_n(\mathcal{F}) > \alpha_d(n)$, then $\mathcal{F}$ contains a copy of $B_d$.

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\]

Find $k$ such that
\[
\frac{1}{\binom{n}{k}} \sum_{S \in \binom{[n]}{k}} h_{n-k}(\mathcal{F}_S) > \alpha_{d-1}(n)
\]

Find $S \in \binom{[n]}{k}$ with $h_{n-k}(\mathcal{F}_S) > \alpha_{d-1}(n)$. 
Extension of Szemerédi’s Cube Lemma: Proof

**Theorem**

If $\mathcal{F} \subseteq 2^{[n]}$ and $h_n(\mathcal{F}) > \alpha_d(n)$, then $\mathcal{F}$ contains a copy of $B_d$.

- $\sum_{k=1}^{n} \frac{1}{\binom{n}{k}} \sum_{S \in \binom{[n]}{k}} h_{n-k}(\mathcal{F}_S) > n\alpha_{d-1}(n)$

- Find $k$ such that
  $$\frac{1}{\binom{n}{k}} \sum_{S \in \binom{[n]}{k}} h_{n-k}(\mathcal{F}_S) > \alpha_{d-1}(n)$$

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If $\mathcal{F} \subseteq 2^{[n]}$ and $h_n(\mathcal{F}) > \alpha_d(n)$, then $\mathcal{F}$ contains a copy of $B_d$.

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- Find $S \in \binom{[n]}{k}$ with $h_{n-k}(\mathcal{F}_S) > \alpha_{d-1}(n)$. 

By induction, $\mathcal{F}_S$ contains a copy of $B_{d-1}$ generated by $X_0, \ldots, X_{d-1}$. 

$\mathcal{F}$ contains a copy of $B_d$ generated by $X_0, \ldots, X_{d-1}$ with $X_d = S$. 

Extension of Szemerédi’s Cube Lemma: Proof

Theorem
If \( \mathcal{F} \subseteq 2^{[n]} \) and \( h_n(\mathcal{F}) > \alpha_d(n) \), then \( \mathcal{F} \) contains a copy of \( \mathcal{B}_d \).

\[
\sum_{k=1}^{n} \frac{1}{\binom{n}{k}} \sum_{S \in \binom{[n]}{k}} h_{n-k}(\mathcal{F}_S) > n\alpha_{d-1}(n)
\]

- Find \( k \) such that
  \[
  \frac{1}{\binom{n}{k}} \sum_{S \in \binom{[n]}{k}} h_{n-k}(\mathcal{F}_S) > \alpha_{d-1}(n)
  \]

- Find \( S \in \binom{[n]}{k} \) with \( h_{n-k}(\mathcal{F}_S) > \alpha_{d-1}(n) \).
- By induction, \( \mathcal{F}_S \) contains a copy of \( \mathcal{B}_{d-1} \) generated by \( X_0, \ldots, X_{d-1} \).
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If \( \mathcal{F} \subseteq 2^{[n]} \) and \( h_n(\mathcal{F}) > \alpha_d(n) \), then \( \mathcal{F} \) contains a copy of \( \mathcal{B}_d \).

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Extension of Szemerédi’s Cube Lemma: Proof

Theorem

If $\mathcal{F} \subseteq 2^{[n]}$ and $h_n(\mathcal{F}) > \alpha_d(n)$, then $\mathcal{F}$ contains a copy of $B_d$.

\begin{itemize}
  \item $\sum_{k=1}^{n} \frac{1}{(n)} \sum_{S \in \binom{n}{k}} h_{n-k}(\mathcal{F}_S) > n\alpha_{d-1}(n)$
  \item Find $k$ such that $\frac{1}{(n)} \sum_{S \in \binom{n}{k}} h_{n-k}(\mathcal{F}_S) > \alpha_{d-1}(n)$
  \item Find $S \in \binom{n}{k}$ with $h_{n-k}(\mathcal{F}_S) > \alpha_{d-1}(n)$.
  \item By induction, $\mathcal{F}_S$ contains a copy of $B_{d-1}$ generated by $X_0, \ldots, X_{d-1}$.
\end{itemize}
Extension of Szemerédi’s Cube Lemma: Proof

**Theorem**
If $\mathcal{F} \subseteq 2^{[n]}$ and $h_n(\mathcal{F}) > \alpha_d(n)$, then $\mathcal{F}$ contains a copy of $B_d$.

- $\sum_{k=1}^{n} \frac{1}{\binom{n}{k}} \sum_{S \in \binom{[n]}{k}} h_{n-k}(\mathcal{F}_S) > n\alpha_{d-1}(n)$

- Find $k$ such that
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- Find $S \in \binom{[n]}{k}$ with $h_{n-k}(\mathcal{F}_S) > \alpha_{d-1}(n)$.

- By induction, $\mathcal{F}_S$ contains a copy of $B_{d-1}$ generated by $X_0, \ldots, X_{d-1}$.

- $\mathcal{F}$ contains a copy of $B_d$ generated by $X_0, \ldots, X_d$ with $X_d = S$. 
Extension of Szemerédi’s Cube Lemma: Proof

Theorem
If $\mathcal{F} \subseteq 2^{[n]}$ and $h_n(\mathcal{F}) > \alpha_d(n)$, then $\mathcal{F}$ contains a copy of $B_d$.

$\sum_{k=1}^{n} \frac{1}{\binom{n}{k}} \sum_{S \in \binom{[n]}{k}} h_{n-k}(\mathcal{F}|_S) > n\alpha_{d-1}(n)$

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$\mathcal{F}$ contains a copy of $B_d$ generated by $X_0, \ldots, X_d$ with $X_d = S$. 
Turán Results

Theorem

If $F \subseteq 2^{[n]}$ and $h_n(F) > \alpha_d(n)$, then $F$ contains a copy of $B_d$. 

Corollary

If $F \subseteq 2^{[n]}$ and $h_n(F) \geq 4n^{1-2d}$, then $F$ contains a copy of $B_d$. 

Partitioning $2^{[n]}$ into consecutive segments of $\sqrt{n}$ levels and applying an averaging argument yields:

Theorem

If $F \subseteq 2^{[n]}$ and $|F| \geq 50n - \frac{1}{2^d}2^{n}$, then $F$ contains a copy of $B_d$. 
Turán Results

**Theorem**
If $F \subseteq 2^{[n]}$ and $h_n(F) > \alpha_d(n)$, then $F$ contains a copy of $B_d$.

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If $F \subseteq 2^{[n]}$ and $h_n(F) \geq 4n^{1 - \frac{2}{2^d}}$, then $F$ contains a copy of $B_d$. 
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If $\mathcal{F} \subseteq 2^{[n]}$ and $h_n(\mathcal{F}) > \alpha_d(n)$, then $\mathcal{F}$ contains a copy of $B_d$.

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If $\mathcal{F} \subseteq 2^{[n]}$ and $h_n(\mathcal{F}) \geq 4n^{1 - \frac{2}{2d}}$, then $\mathcal{F}$ contains a copy of $B_d$.

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If $\mathcal{F} \subseteq 2^{[n]}$ and $h_n(\mathcal{F}) > \alpha_d(n)$, then $\mathcal{F}$ contains a copy of $B_d$.

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If $\mathcal{F} \subseteq 2^{[n]}$ and $h_n(\mathcal{F}) \geq 4n^{1-\frac{2}{2d}}$, then $\mathcal{F}$ contains a copy of $B_d$.

- Partitioning $2^{[n]}$ into consecutive segments of $\sqrt{n}$ levels and applying an averaging argument yields:

Theorem
If $\mathcal{F} \subseteq 2^{[n]}$ and $|\mathcal{F}| \geq 50n^{-1/2^d} \cdot 2^n$, then $\mathcal{F}$ contains a copy of $B_d$. 
Ramsey Problem

- How many parts are needed to partition $2^n$ into $\mathcal{B}_d$-free families?

Prior Work

Theorem $r(n, d) \geq \frac{1}{16} \cdot n^2$
Ramsey Problem

- How many parts are needed to partition $2^{[n]}$ into $\mathcal{B}_d$-free families?
- Let $r(n, d)$ be the minimum number of parts needed.
Ramsey Problem

- How many parts are needed to partition $2^n$ into $\mathcal{B}_d$-free families?
- Let $r(n, d)$ be the minimum number of parts needed.

Prior Work

- Clearly, $r(n, 1) = n + 1$. 

Thank You.
Ramsey Problem

- How many parts are needed to partition $2^{|n|}$ into $B_d$-free families?
- Let $r(n, d)$ be the minimum number of parts needed.

Prior Work

- Clearly, $r(n, 1) = n + 1$.
- [Gunderson–Rödl–Sidorenko 1999] For $n$ sufficiently large

\[
(1 - o(1)) \frac{3}{4} \cdot n^{1/2} \leq r(n, 2) \leq (1 + o(1)) \cdot n^{1/2}.
\]
Ramsey Problem

- How many parts are needed to partition $2^{[n]}$ into $B_d$-free families?
- Let $r(n, d)$ be the minimum number of parts needed.

Prior Work

- [Gunderson–Rödl–Sidorenko 1999] For $d > 2$, there exists $c_d$ such that for $n$ sufficiently large
  \[ c_d \cdot n^{\frac{1}{2^d}} \leq r(n, d) \leq n^{\frac{d}{2^d - 1}(1+o(1))}. \]
Ramsey Problem

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$$c_d \cdot n^{\frac{1}{2^d}} \leq r(n, d) \leq n^{\frac{d}{2^d-1}}(1+o(1)).$$

- Here, $c_d = (10d)^{-d}(1 + o(1))$. 
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Theorem

\[ r(n, d) \geq \frac{1}{4} \cdot n^{\frac{2}{2d}} \]
Ramsey Problem

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\[
c_d \cdot n^{\frac{1}{2d}} \leq r(n, d) \leq n^{\frac{d}{2d - 1}(1 + o(1))}.
\]

- Here, $c_d = (10d)^{-d}(1 + o(1))$.

Theorem

\[
r(n, d) \geq \frac{1}{4} \cdot n^{\frac{2}{2d}}
\]

Thank You.