Topology 2, Math 681, Spring 2020: Notes

Krzysztof Chris Ciesielski

Classes of Tuesday and Thursday, January 14 and 16:

- Go briefly over syllabus
- Hand homework: last semester final test exercises, due on January 21
- The last semester's abbreviated notes are still available on http://www.math.wvu.edu/~kciesiel/teach/Fall2019/Fall2019.html

Quick Review (Also definition of topological space and continuous maps)

- For $\mathcal{B} \subset \mathcal{P}(X)$ let
$$\mathcal{T}(\mathcal{B}) := \{U \subset X: \forall x \in U \exists B \in \mathcal{B}(x \in B \subset U)\}.$$
If $\mathcal{B}$ satisfies

  (B1) For every $x \in X$ there is a $B \in \mathcal{B}$ with $x \in B$ (i.e., $\bigcup \mathcal{B} = X$).

  (B2) For every $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$ there is a $B \in \mathcal{B}$ with $x \in B \subset B_1 \cap B_2$.

  then $\mathcal{T}(\mathcal{B})$ is a topology on X. The family $\mathcal{B}$ is a basis for the topology $\mathcal{T}(\mathcal{B})$.
- X is *Hausdorff* (or a T_2 *space*) provided for every distinct $x, y \in X$ there exists disjoint open sets $U, V \subset X$ such that $x \in U$ and $y \in V$.
- If X is a Hausdorff topological space, then any sequence $\langle x_n \rangle_{n=1}^{\infty}$ of points of X *converges* to at most one point in X.
- The product of two Hausdorff topological spaces is a Hausdorff space. A subspace of a Hausdorff topological space is a Hausdorff space.
- A function $f: X \to Y$ is *continuous* provided $f^{-1}(V)$ is open in X for every open subset V of Y.
- If $\mathcal{B}$ a basis for a topological space Y, then $f: X \to Y$ is continuous if, and only if, $f^{-1}(B)$ is open in X for every $B \in \mathcal{B}$.
- *Product topology* $\mathcal{T}_{prod}$ on $X = \prod_{\alpha \in J} X_\alpha$ is generated by subbasis $\mathcal{S}_{prod} = \{\pi_\beta^{-1}(U_\beta)$ for all $\beta \in J$ and open subsets U_β of $X_\beta\}$

Class of Tuesday, January 21:

Continue Review

- *Product topology* $\mathcal{T}_{prod}$ on $X = \prod_{\alpha \in J} X_\alpha$ is generated by subbasis $\mathcal{S}_{prod} = \{\pi_\beta^{-1}(U_\beta)$ for all $\beta \in J$ and open subsets U_β of $X_\beta\}$
- *If $f_\alpha \colon A \to X_\alpha$ and $f \colon A \to X$ is given by $f(a)(\alpha) = f_\alpha(a)$, then*

  continuity of f implies the continuity of each f_α;

  continuity of all f_α's implies the continuity of $f \colon A \to \langle X, \mathcal{T}_{prod} \rangle$;
- A *metric space* is a pair $\langle X, d \rangle$, where d is a metric on X.
  $\mathcal{B}_d = \{B(x, \varepsilon) \colon x \in X \ \& \ \varepsilon > 0\}$ is a basis for a topology on X.
- $\mathcal{T}(\mathcal{B}_d)$ is the metric topology on X (for metric d).
- Subspace of a metric space is metric.
- Every metrizable space is Hausdorff.
- Finite and countable product of metric spaces is metrizable.
- A topological space $\langle X, \mathcal{T} \rangle$ is *first countable* provided for every $x \in X$ there exists a countable basis $\mathcal{B}_x$ of X at x;
- *Let X be a first countable space and let $A \subset X$. Then $x \in \mathrm{cl}(A)$ if, and only if, there is a sequence of points of A converging to x.*
- A topological space X is *connected* provided it **does not** exist a pair U, V of open, non-empty disjoint sets with $X = U \cup V$.
- **(Star Lemma)** Let $\{A_\alpha\}_{\alpha \in J}$ be a family of connected subspaces of X. If $\bigcap_{\alpha \in J} A_\alpha \neq \emptyset$, then $\bigcup_{\alpha \in J} A_\alpha$ is connected.
- A closure of a connected space is connected.
- Finite product of connected spaces is connected: sketch proof.
- $\mathbb{R}^\omega$ with the product topology is connected: sketch proof.
- Continuous image of connected space is connected: sketch proof.
- $A \subset \mathbb{R}$ is connected if, and only if, A is convex (an interval).
- Intermediate Value Theorem.

Class of Thursday, January 23:

Administer Quiz #0, review.

Finish Review

- Definition of *path connectedness*.
- *Topologists sine curve*: it is connected but not path connected.
- Show that every continuous function $f\colon [0,1] \to [0,1]$ has a fixed point.

Sections 26 and part of 27, with mixed order: compactness

Definition 1 Let Y be a subset of a topological space X. A family $\mathcal{U}$ of subsets of X is a *covering* of Y provided $Y \subset \bigcup \mathcal{U}$. A covering $\mathcal{U}$ of Y is an *open covering* of Y provided every $U \in \mathcal{U}$ is open in X.

Definition 2 A topological space X is *compact* provided for every open cover $\mathcal{U}$ of X there exists a finite subfamily $\mathcal{U}_0$ of $\mathcal{U}$ that covers X (i.e., $\mathcal{U}_0 \subset \mathcal{U}$ is finite and $X = \bigcup \mathcal{U}_0$). Such a family $\mathcal{U}_0$ will be referred to as a (finite) *subcover* of $\mathcal{U}$.

Note: Although subcover $\mathcal{U}_0$ of $\mathcal{U}$ is defined in term of a union $\bigcup \mathcal{U}_0$, this union usually does not belong to $\mathcal{U}$!

Go over Examples 1 and 4: Neither $\mathbb{R}$ nor $(0,1]$ are compact.

Go over Examples 2 and 3: Every finite space X is compact. So is $X = \{L\} \cup \{a_n \colon n = 1,2,3,\ldots\} \subset \mathbb{R}$, provided $\lim_n a_n = L$.

Lemma 1 (Lemma 26.1)

Class of Tuesday, January 28:

Return Quiz #0.

Recall

- X is *compact* provided for every open cover $\mathcal{U}$ of X contains a finite subcover $\mathcal{U}_0$ of $\mathcal{U}$ that covers X.

New material

Theorem 2 (Theorem 26.2) *Closed subspace of compact space is compact.*

Theorem 3 (Theorem 26.3; Important!) *Every compact subspace of a Hausdorff space is closed.*

Go over Example 6 of a T_1 space X and compact $A \subset X$ such that $\mathrm{cl}(A)$ is not compact:
$X = \mathbb{R}$, $\mathcal{T} = \{\emptyset\} \cup \{U \subset \mathbb{R} : \mathbb{N} \setminus U \text{ if finite}\}$, and $A = \mathbb{N}$.
$\mathrm{cl}(\mathbb{N}) = \mathbb{R}$ and $\mathcal{U} = \{N \cup \{x\} : x \in \mathbb{R}\}$ is its open cover with no finite subcover.

Proof of the theorem is based on:

Lemma 4 (Lemma 26.4) *Let X be Hausdorff. For every compact subspace Y of X and every $x \in X \setminus Y$ there exists disjoint open sets U and V in X such that $x \in U$ and $Y \subset V$.*

Written assignment for Thursday, January 30: Exercise 5 p. 171: *Let A and B be disjoint compact subspaces of the Hausdorff space Y. Show, that there exist disjoint open sets U and V containing A and B, respectively.*

Theorem 5 (Theorem 27.1) *Every closed interval $[a, b]$ in $\mathbb{R}$ is compact.*

Try to go over the proof this class.

Corollary 6 (Corollary 27.3 for $\mathbb{R}$) *A subspace X of $\mathbb{R}$ is compact if, and only if, it is closed and bounded.*

Theorem 7 (Thm 26.5) *Continuous image of a compact space is compact.*

Theorem 8 (Theorem 26.6) *If $f : X \to Y$ is a continuous bijection, X is compact, and Y is Hausdorff, then f is a homeomorphism.*

Corollary 9 (Thm 27.4: Extreme Value Theorem) *For every compact space X and continuous function $f : X \to \mathbb{R}$ there exist $c, d \in X$ such that $f(c) \leq f(x) \leq f(d)$ for every $x \in X$. In particular, this holds for $X = [a, b]$.*

Go over Exercises 1a, 4, and 6 from section 26.

Class of Thursday, January 30:

Collect homework. Be ready for Quiz #1 next class.

Recall

- X is *compact* provided for every open cover $\mathcal{U}$ of X contains a finite subcover $\mathcal{U}_0$ of $\mathcal{U}$ that covers X.
- Closed subspace of compact space is compact.
- Every compact subspace of a Hausdorff space is closed.
- For every compact subspace Y of a Hausdorff space X and every $x \in X \setminus Y$ there are disjoint open U and V such that $x \in U$ and $Y \subset V$.
- Every closed interval $[a, b]$ in $\mathbb{R}$ is compact (not proved yet).
- A subspace X of $\mathbb{R}$ is compact if, and only if, it is closed and bounded.
- Continuous image of a compact space is compact.

New material

Prove that: *Every closed interval* $[a, b]$ *in* $\mathbb{R}$ *is compact.*

Theorem 10 (Thm 26.7) *Finite product of compact spaces is compact.*

Remark: Actually, arbitrary product of compact spaces is compact. This is Tychonoff Theorem. But its proof is more difficult.

Proof of Theorem 10 based on **very important**

Lemma 11 (Lem 26.8: The Tube Lemma) *Let* Y *be compact and* $x \in X$. *If an open set* W *of* $X \times Y$ *contains* $\{x\} \times Y$, *then there is an open set* U *in* X *such that* $\{x\} \times Y \subset U \times Y \subset W$.

Corollary 12 (Corollary 27.3 for $\mathbb{R}^n$) *A subspace* X *of* $\mathbb{R}^n$ *is compact if, and only if, it is closed and bounded.*

Corollary 13 (Extreme Value Theorem for $\mathbb{R}^n$) *If* R *is a closed bounded subset of* $\mathbb{R}^n$, *then for every continuous function* $f\colon R \to \mathbb{R}$ *there exist* $c, d \in R$ *such that* $f(c) \leq f(x) \leq f(d)$ *for every* $x \in R$.

Go over of Exercise 7 section 26.

Class of Tuesday, February 4:

Administer Quiz #1.
Return graded Homework # 1with a typeset solution.
Recall

- **The Tube Lemma:** Let Y be compact and $x \in X$. If an open set W of $X \times Y$ contains $\{x\} \times Y$, then there is an open set U in X such that $\{x\} \times Y \subset U \times Y \subset W$.
- Finite product of compact spaces is compact.
- **(Extreme Value Theorem for $\mathbb{R}^n$)** If R is a closed bounded subset of $\mathbb{R}^n$, then for every continuous function $f\colon R \to \mathbb{R}$ there exist $c, d \in R$ such that $f(c) \leq f(x) \leq f(d)$ for every $x \in R$.

Go over Exercise 1b from section 26.

Go over Exercise 4 section 27.

Definition 3 A collection $\mathcal{C}$ of subsets of X has *finite intersection property, fip,* provided $\bigcap \mathcal{C}_0 \neq \emptyset$ for every finite $\mathcal{C}_0 \subset \mathcal{C}$.

Theorem 14 (Thm 26.9) *X is compact if, and only if, $\bigcap \mathcal{C} \neq \emptyset$ for every family $\mathcal{C}$ of closed subsets of X having fip.*

Written assignment for Thursday, February 6: Ex. 8 and 9 p. 171.

Definition 4 A function f from a metric space $\langle X, d\rangle$ into a metric space $\langle Y, \rho\rangle$ is said to be *uniformly continuous* provided for every $\varepsilon > 0$ there is a $\delta > 0$ such that for every $x_1, x_2 \in X$

$$d(x_0, x_1) < \delta \Longrightarrow \rho(f(x_0), f(x_1)) < \varepsilon.$$

Theorem 15 (Thm 27.6, Uniform continuity theorem) *Let f be a continuous function from a compact metric space $\langle X, d\rangle$ into a metric space $\langle Y, \rho\rangle$. Then f is uniformly continuous.*

Prove using the Lebesgue number lemma, where a *diameter* of a subset D of a metric space $\langle X, d\rangle$ is defined as $\operatorname{diam}(D) = \sup\{d(x, y)\colon x, y \in D\}$.

Lemma 16 (Lem 27.5, the Lebesgue number lemma) *Let $\mathcal{A}$ be an open cover of a metric space $\langle X, d\rangle$. If X is compact, then there exists a $\delta > 0$, known as a* **Lebesgue number**, *such that for every $D \subset X$ of diameter $< \delta$, there exists an $A \in \mathcal{A}$ with $D \subset A$.*

Steps of the proof:

- For a metric space $\langle X, d\rangle$, $x \in X$, and non-empty $A \subset X$ we define the *distance from x to A* as $d(x, A) = \inf\{d(x, a) \colon a \in A\}$.
- Show that $X \ni x \mapsto d(x, A) \in \mathbb{R}$ is continuous.
- Prove the Lebesgue number lemma.
- Prove the uniform continuity theorem.

Class of February 6:

Collect homework. Return Quiz. Explain its question # 3.
Recall that:

- A collection $\mathcal{C}$ of subsets of X has *finite intersection property, fip,* provided $\bigcap \mathcal{C}_0 \neq \emptyset$ for every finite $\mathcal{C}_0 \subset \mathcal{C}$.
- X is compact if, and only if, $\bigcap \mathcal{C} \neq \emptyset$ for every family $\mathcal{C}$ of closed subsets of X having fip.
- $f\colon \langle X, d\rangle \to \langle Y, \rho\rangle$ is *uniformly continuous* provided for every $\varepsilon > 0$ there is a $\delta > 0$ such that $\rho(f(x_0), f(x_1)) < \varepsilon$ for every $x_1, x_2 \in X$ with $d(x_0, x_1) < \delta$.
- **Uniform continuity theorem:** Every continuous function from a compact metric space into a metric space is uniformly continuous.
- A *diameter* of a subset D of a metric space $\langle X, d\rangle$ is defined as the number $\operatorname{diam}(D) = \sup\{d(x, y)\colon x, y \in D\}$.
- **Lebesgue number lemma:** For every open cover $\mathcal{A}$ of a metric space $\langle X, d\rangle$ there is a $\delta > 0$, *Lebesgue number*, such that for every $D \subset X$ of diameter $< \delta$, there is an $A \in \mathcal{A}$ with $D \subset A$.

New material

Go over Exercise 5 page 178. This is Baire category theorem.

Definition 5 A point x in a topological space X is an *isolated point* provided $\{x\}$ is open in X.

Theorem 17 (Thm 27.7) *Let X be a compact Hausdorff space. If X has no isolated points, then X is uncountable.*

Deduce this from the Baire category theorem, Ex 5
Solve Exercise 4 page 178.
Students, try to solve Ex 11 page 171 (not homework).

Section 28: Limit Point Compactness

Definition 6 A space X is *limit point compact* provided every infinite subset of X has a limit point.

Theorem 18 (Thm 28.1) *If X is compact, then X is limit point compact, but not conversely.*

Go over Examples 1 and 2.

Definition 7 A space X is *sequentially compact* provided every sequence in X has a convergent subsequence.

Theorem 19 (Thm 28.2) *For a metrizable space X, the following are equivalent:*

(1) X is compact.

(2) X is limit point compact.

(3) X is sequentially compact.

Prooved "(1) implies (2)" and "(2) implies (3)." The remaining implication will be proved next class.

Remark: "(2) implies (3)" requires only first countability of X.

Class of February 11:

Collect homework. Be ready for a Quiz next class.
Recall that:

- X is: *limit point compact* provided every infinite subset of X has a limit point; *sequentially compact* provided every sequence in X has a convergent subsequence.
- **Thm** For metrizable spaces, the three notions, *compactness*, *limit point compactness*, and *sequential compactness*, are equivalent.

New material

Finish the proof of the theorem showing that, for metric spaces, sequential compactness implies compactness.

Mention Example 3: the space satisfies (1)–(3), but is not first countable, so not metrizable.

Go over Exercises 1 and 2 page 181.

Students, look at Exercise 7 pages 181–182 (not homework).

Section 29: Local Compactness

Definition 8 A space X is *locally compact* provided every $x \in X$ there is an open set $U \ni x$ such that $\mathrm{cl}(U)$ is compact.

Compact implies locally compact.
Go over Examples 1 and 2.
$\mathbb{Q}$ is not locally compact: Exercise 1 page 186.
State and prove (sketch only) Theorem 29.1.
Define *one-point compactification* of a locally compact space.
Go over Example 4.

Time permitting

State Theorem 29.2, Corollary 29.3, and Corollary 29.4.

Class of February 13:

Administer Quiz #2. Return homework.

Section 30: The Countability Axioms

The next definition and theorem were covered last semester.

Definition 9 A topological space X is *first countable* (or *satisfies the first countability axiom*) provided for every $x \in X$ there exists a countable basis $\mathcal{B}_x$ of X at x.

Theorem 20 *Let X be a first countable topological space and let $A \subset X$. Then $x \in \text{cl}(A)$ if, and only if, there is a sequence of points of A converging to x. Moreover, the implication "$\Longleftarrow$" does not require the assumption of first countability.*

New material:

Definition 10 A topological space X is *second countable* (or *satisfies the second countability axiom*) provided X has a countable basis.

Go over Examples 1 and 2.
Go over Theorem 30.2.

Definition 11

- A subset A of a space X is *dense (in X)* provided $\text{cl}(A) = X$.
- A topological space X is *separable* provided X has a countable dense subset D, that is, such that $\text{cl}(D) = X$.
- A topological space X is *Lindelöf* provided every open cover of X has a countable subcover.

Written assignment for Tuesday, February 18: Exercises 11 and 13, page 194.

Go over Theorem 30.3.

Go over Examples 3 and 4. (Very important!)

Class of February 18:

Collect homework. Return Quiz #2.

Recall that:

- A topological space X is: *second countable* provided X has a countable basis; *separable* provided X has a countable dense subset; *Lindelöf* provided every open cover of X has a countable subcover.
- Second countability implies: first countability, separability, and Lindelöf property. None of these implications can be reversed, as proved by $\mathbb{R}_\ell$.
- $\mathbb{R}_\ell$ is Lindelöf; main steps:
  - Note, that it is enough to consider only the covers $\{[a_\xi, b_\xi)\}_{\xi\in J}$.
  - $\mathcal{U} = \{(a_\xi, b_\xi)\}_{\xi\in J}$ is an open cover of $C = \bigcup_{\xi\in J}(a_\xi, b_\xi)$.
  - We can find countable $J_0 \subset J$ with $C = \bigcup_{\xi\in J_0}(a_\xi, b_\xi)$.
  - $\mathbb{R}\backslash C$ is countable. Find countable $J_1 \subset J$ with $\mathbb{R}\backslash C \subset \bigcup_{\xi\in J_1}[a_\xi, b_\xi)$.
  - $\mathcal{V}_0 = \{[a_\xi, b_\xi)\}_{\xi\in J_0\cup J_1} \subset \mathcal{V}$ is countable and covers $\mathbb{R} = C\cup(\mathbb{R}\backslash C)$.
- Product of Lindelöf spaces need not be Lindelöf, as proved by $\mathbb{R}_\ell$.

New material:

Solve Exercise 10, page 194: *Show that a countable product of separable spaces is separable.*

Discuss the table

	subspace	closed subspace	countable product	arbitrary product	continuous image
compact	N, $[0,1)$; $[0,1]$	Y	Y, Tych. Thm	Y, Tych. Thm	Y
connected	N, $\mathbb{R}\setminus(0,1)$	N, $\mathbb{R}\setminus(0,1)$	Y	Y	Y
Hausdorff	Y	Y	Y	Y	N, Ex a
2nd count	Y	Y	Y	N, $\mathbb{R}^{\text{uncount}}$ p. 133	N, Ex c
1st count	Y	Y	Y	N, $\mathbb{R}^{\text{uncount}}$ p. 133	N, Ex b,c
separable	N, $L \subset (\mathbb{R}_\ell)^2$	N, $L \subset (\mathbb{R}_\ell)^2$	Y, 10 p. 194	N, 16 p. 195	Y, 11 p. 194
Lindelöf	N, $\mathbb{R}_\infty$	Y, 9 p. 194	N, $(\mathbb{R}_\ell)^2$	N, $(\mathbb{R}_\ell)^2$	Y, 11 p. 194
metrizable	Y	Y	Y	N, $\mathbb{R}^{\text{uncount}}$ p. 133	N, Ex a

Here the space X_∞, in particular $\mathbb{R}_\infty$, is the one point compactification of a discrete space X, that is, $X_\infty = X \cup \{\infty\}$, where $\infty \notin X$, has the topology $\tau = \mathcal{P}(X) \cup \{X_\infty \setminus F \colon F \text{ is a finite subset of } X\}$.

Example. For a set X let τ_d be a discrete topology on X and $\mathcal{T}$ an arbitrary topology on X. Then a function $f \colon \langle X, \tau_d\rangle \to \langle X, \mathcal{T}\rangle$, given by $f(x) = x$, is continuous bijection.

(a) If $\mathcal{T} = \{\emptyset, X\}$ is anti-discrete topology and $X = \mathbb{N}$, then domain of f is metric, while $f[X]$ is not Hausdorff.

(b) If $X = \mathbb{R}^\omega$ and $\mathcal{T}$ is a box topology, then domain of f is first countable (as metric), while $f[X]$ is not first countable.

(c) Let $X = \mathbb{N}$ and $\mathcal{T}$ be such that $\langle X, \mathcal{T}\rangle$ is not second countable. (This will be proved together with Tychonoff's theorem.) Then domain of f is second countable, while $f[X]$ is not (hence, it is not first countable).

Suggested Exercises to examine by the students: 1, 3, and 12, page 194.

Class of February 20:

Return homework.

Go over **Exercises 2, 4, and 5, page 194**; 16, page 195.
Start new section:

Section 31: The Separation Axioms

- (already seen) X is a T_0 *space* provided for every distinct $x, y \in X$ there exists an open set $U \subset X$ such that either $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$ (i.e., such that U contains precisely one of the points x and y).
- (already seen) X is a T_1 *space* provided for every distinct $x, y \in X$ there exists an open set $U \subset X$ such that $x \in U$ and $y \notin U$.
- (already seen) X is *Hausdorff* (or a T_2 *space*) provided for every distinct $x, y \in X$ there exist disjoint open sets $U, V \subset X$ such that $x \in U$ and $y \in V$.
- (new) X is *regular* (or a T_3 *space*) provided it is a T_1 space and for every closed set K in X and $x \in X \setminus K$ there exist disjoint open sets $U, V \subset X$ such that $x \in U$ and $K \subset V$.
- (new) X is *normal* (or a T_4 *space*) provided it is a T_1 space and for every disjoint closed sets K and L in X there exist disjoint open sets $U, V \subset X$ such that $K \subset U$ and $L \subset V$.

Go over Lemma 31.1.
Go over Exercises 1 and 2.
Go over Theorem 31.2: a subspace of regular space is regular; the product of regular spaces is regular. (Same for Hausdorff spaces, as proved last semester.) Note that it is false for the normal spaces. (Point to where the subspace part of the proof for the regular spaces brakes for the normal spaces.)

Class of February 25:

Recall

- X is *regular* (or a T_3 *space*) provided it is a T_1 space and: for every closed set K in X and $x \in X \setminus K$ there exist disjoint open sets $U, V \subset X$ such that $x \in U$ and $K \subset V$ (equivalently, for every open U and $x \in U$, there an exists open V with $x \in V \subset \mathrm{cl}(V) \subset U$).
- X is *normal* (or a T_4 *space*) provided it is a T_1 space and: for every disjoint closed sets K and L in X there exist disjoint open sets $U, V \subset X$ such that $K \subset U$ and $L \subset V$ (equivalently, for every open U and closed $F \subset U$, there exists an open V with $F \subset V \subset \mathrm{cl}(V) \subset U$).
- Theorem 31.2: a subspace of regular space is regular; the product of regular spaces is regular.

Go over Example 1: $\mathbb{R}_K$ is Hausdorff but not regular.
Go over Exercise 4.
Go over Example 2: $\mathbb{R}_\ell$ is normal.
Go over Theorem 7.8 page 80:

There is no injection $f\colon \mathcal{P}(A) \to A$ *or surjection* $f\colon A \to \mathcal{P}(A)$.

(To be proved next class.)

Use it, in Example 3, to show that $(\mathbb{R}_\ell)^2$ is not normal.

Note that the product of normal spaces need not be normal. Also, $(\mathbb{R}_\ell)^2$ is regular but not normal.

Latter we will prove that $(\mathbb{R}_\ell)^2$ is homeomorphic to a subspace of some normal spaces. So, a subspace of normal space need not be normal.

Written assignment for Thursday, February 27: Exercise 5, page 199: *Let* $f, g\colon X \to Y$ *be continuous; assume that* Y *is Hausdorff. Show that the set* $Z = \{x \in X\colon f(x) = g(x)\}$ *is closed in* X*. (Do not assume on* Y *anything except being Hausdorff!)*

Class of February 27:

Collect homework.

Recall

- $\mathbb{R}_K$ is Hausdorff but not regular.
- $\mathbb{R}_\ell$ is normal.
- $(\mathbb{R}_\ell)^2$ is not normal (but regular). So, product of normal spaces need not be normal.

New material:

Go over Theorem 7.8 page 80:

There is no injection $f\colon \mathcal{P}(A) \to A$ or surjection $f\colon A \to \mathcal{P}(A)$.

Section 32: Normal spaces

Thm 32.2: Every metrizable space is normal.

Thm 32.3: Every compact Hausdorff space is normal.

Show that every regular Lindelöf space is normal. This is Ex 4 page 205. Proof the same as for Thm 32.1.

Corollary: the product of two Lindelöf spaces need not be Lindelöf, justified by $(\mathbb{R}_\ell)^2$.

Go (briefly) over Example 1.

Go over Exercises 1, 2, 3, 5.

Section 33: The Urysohn Lemma

State (proof next meeting):

Urysohn Lemma: *If X is normal and A and B are closed disjoint subsets of X, then there exists continuous $f\colon X \to [0,1]$ such that $f[A] \subseteq \{0\}$ and $f[B] \subseteq \{1\}$.*

Define *completely regular* (or $T_{3.5}$) spaces.

Stated Thm 33.2: Subspace of a completely regular space is completely regular. Product of completely regular spaces is completely regular.

Class of March 3:

Mid term test will be in a week: Tuesday, March 10, in class.
Recall

- Every regular Lindelöf space is normal.
- Stated **Urysohn Lemma:**: *If X is normal and A and B are closed disjoint subsets of X, then there exists continuous $f\colon X \to [0,1]$ such that $f[A] \subseteq \{0\}$ and $f[B] \subseteq \{1\}$.*
- X is *completely regular* (or a $T_{3.5}$ *space*) when it is a T_1 space and for every closed $K \subset X$ and $x \in X \setminus K$ there is continuous $f\colon X \to [0,1]$ s.t. $f[K] \subseteq \{0\}$ and $f(x) = 1$.
- Noticed that normality implies complete regularity and complete regularity implies regularity.
- Stated Thm 33.2: Subspace of a completely regular space is completely regular. Product of completely regular spaces is completely regular.

New material:
Prove **Urysohn Lemma.**
Prove Thm 33.2.

Go over a part of Exercise 4, page 213:

(i) *If $f\colon X \to [0,1]$ is continuous, then $A = f^{-1}(0)$ is a G_δ-set (that is, A is an intersection of countably many open sets).*

Written assignment for Tuesday, March 5: A more difficult direction of Exercise 4, page 213:

(a) Prove that if X is normal, then for every closed G_δ set $A \subset X$ there is a continuous $f\colon X \to [0,1]$ such that $f^{-1}(0) = A$.

Exercise 5, page 213 (a version of Urysohn Lemma): Let X be normal. *There exists a continuous $f\colon X \to [0,1]$ such that $f^{-1}(0) = A$ and $f^{-1}(1) = B$ if, and only if, A and B are disjoint closed G_δ sets.*

PROOF. "$\Longrightarrow$" follows from (i).

"$\Longleftarrow$" By (a) there exists continuous functions $f_A, f_B\colon X \to [0,1]$ with $f_A^{-1}(0) = A$ and $f_B^{-1}(0) = B$. Then $f(x) = \frac{f_A(x)}{f_A(x)+f_B(x)}$ is as needed. ∎

Suggestion: Look over Exercises 1 and 3, page 212; 7 and 8, page 213.

Class of March 5:

Collect homework.
Mid term test will be in a week: Tuesday, March 10, in class.
Review for the test.

Go over the expanded table (raws regular, completely regular, normal):

	subspace	closed subspace	countable product	arbitrary product	continuous image
2nd countable	Y	Y	Y	N, $\mathbb{R}^{\text{uncountable}}$	N, Ex c
1st countable	Y	Y	Y	N, $\mathbb{R}^{\text{uncountable}}$	N, Ex b,c
separable	N, $L \subset (\mathbb{R}_\ell)^2$	N, $L \subset (\mathbb{R}_\ell)^2$	Y, 10 p. 194	N, 16 p. 195	Y, 11 p. 194
Lindelöf	N, $\mathbb{R}_\infty$	Y	N, $(\mathbb{R}_\ell)^2$	N, $(\mathbb{R}_\ell)^2$	Y, 11 p. 194
compact	N, $[0,1); [0,1]$	Y	Y, Tych. Thm	Y, Tych. Thm	Y
connected	N, $\mathbb{R} \setminus (0,1)$	N, $\mathbb{R} \setminus (0,1)$	Y	Y	Y
Hausdorff	Y	Y	Y	Y	N, Ex a
regular	Y	Y	Y	Y	N, Ex a
completely reg	Y	Y	Y	Y	N, Ex a
normal	N, p. 203	Y, 1 p. 205	N, $(\mathbb{R}_\ell)^2$	N, $(\mathbb{R}_\ell)^2$	N, Ex a
metrizable	Y	Y	Y	N, $\mathbb{R}^{\text{uncountable}}$	N, Ex a

Answers

Example. For a set X let τ_d be a discrete topology on X and $\mathcal{T}$ an arbitrary topology on X. Then a function $f \colon \langle X, \tau_d \rangle \to \langle X, \mathcal{T} \rangle$, given by $f(x) = x$, is continuous bijection.

(a) If $\mathcal{T} = \{\emptyset, X\}$ is anti-discrete topology and $X = \mathbb{N}$, then domain of f is metric, while $f[X]$ is not Hausdorff.

(b) If $X = \mathbb{R}^\omega$ and $\mathcal{T}$ is a box topology, then domain of f is first countable (as metric), while $f[X]$ is not first countable.

(c) Let $X = \mathbb{N}$ and $\mathcal{T}$ be such that $\langle X, \mathcal{T} \rangle$ is not second countable. (This will be proved together with Tychonoff's theorem.) Then domain of f is second countable, while $f[X]$ is not (since it is not first countable).

Be ready to cite the following main theorems, not proved yet.

- **Urysohn metrization theorem:** *Every regular second countable space X is metrizable.*
- **Tietze Extension Theorem:** *f X is normal, $K \subset X$ is closed, and $f: K \to [0,1]$ is continuous, then f can be extended to a continuous $F: X \to [0,1]$.*
- **Tychonoff Theorem:** *Arbitrary product of compact spaces is compact.*

Solve the following exercises:
Ex 11 p. 171.
Ex 8, 9, 13, 15 p. 19.
Ex 7 p. 213.

Class of March 10:
Administer mid term test.

Class of March 12:

Collect mid term bonus exercise solutions.
Return, together with solutions: (1) last homework; (b) mid term test.
Discuss results of the mid term test and go over the solutions.

Time permitting: start proving

Theorem 21 (Urysohn metrization theorem) *Every regular second countable space X is metrizable.*

PROOF. (Sketch)

1. Notice that every regular second countable space is normal (as it is regular Lindelöf), so we can use Urysohn Lemma.

2. Prove that there exists a countable family $\mathcal{F}$ of continuous functions $f\colon X \to [0,1]$ such that:

   (∗) For every open $U \subset X$ and $x \in U$ there is an $f \in \mathcal{F}$ such that $f(x) > 0$ and $f[X \setminus U] \subset \{0\}$.

   A family $\mathcal{F}$ of continuous functions $f\colon X \to \mathbb{R}$ satisfying (∗) is said to *separate points from closed sets* in X.

3. Prove that (Thm 34.2): *For any T_1 space X family $\{f_\alpha\}_{\alpha\in J}$ separating points from closed sets in X the mapping $F\colon X \to \mathbb{R}^J$, $F(x)(\alpha) = f_\alpha(x)$, is an imbedding.*

4. Notice that, by 1 and 2, our regular second countable space X can be imbedded into $\mathbb{R}^\omega$. Since $\mathbb{R}^\omega$ is metrizable, so is X.