MATH 251
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SAMPLE TEST # 4

Solve the following exercises. **Show your work.**

Ex. 1. Set up the integral formulas, **including the limits of the integrations,** for the following problems. *Do not evaluate the integrals!* Where appropriate, *use polar, cylindrical, or spherical coordinates.*

(a) The volume of the solid bounded by $z = x^2 + y^2$, $z = 0$, $x = 0$, $y = 0$, and $x + y = 1$.

Solution: If T is a triangle bounded by $x = 0$, $y = 0$, and $x + y = 1$ (i.e., $y = 1 - x$), then $V = \int\int\int_E 1 dV = \int\int_T \int_0^{x^2+y^2} 1\, dz\, dA = \int_0^1 \int_0^{1-x} \int_0^{x^2+y^2} 1\, dz\, dy\, dx$

(b) The mass of the plane lamina bounded by $y = x^2$ and $y = 2x + 3$, with the density $\delta(x, y) = x^2$.

Solution: If $y = x^2$ and $y = 2x + 3$, then $x^2 = 2x + 3$, that is, $x^2 - 2x - 3 = 0$, so that $x = -3$ and $x = 1$. Then $mass = \int\int_R \delta(x, y) dA = \int_{-3}^1 \int_{x^2}^{2x+3} x^2\, dy\, dx$.

(c) The mass of the solid T with the density $\delta(x, y, z) = x^2 + e^z$ bounded by the surfaces: $6x + 2y + z = 12$, $x = 0$, $y = 0$, and $z = 0$.

Solution: The solid is a tetrahedron with a triangular base B on the xy-plane $z = 0$ bounded by $6x + 2y = 12$, $x = 0$, $y = 0$. The upper bound of T is $z = 12 - 6x - 2y$. So, $mass = \int\int\int_T \delta(x, y, z)\, dV = \int\int_B \int_0^{12-6x-2y} (x^2 + e^z)\, dz\, dA$.

Since the triangle side $6x + 2y = 12$ means that $y = 6 - 3x$, which quals 0 for $x = 2$, we get $mass = \int_0^2 \int_0^{6-3x} \int_0^{12-6x-2y} (x^2 + e^z)\, dz\, dy\, dx$.

Ex. 2. Evaluate the integrals:

(a) $\displaystyle\int_0^1 \int_0^{\pi} \frac{1}{x+1} + \sin y\; dy\; dx =$

Solution: $int = \int_0^1 \left[\frac{1}{x+1} y - \cos y\right]_0^{\pi} dx = \int_0^1 \left(\frac{1}{x+1}\pi - (\cos\pi - \cos 0)\right) dx$. So

$int = \int_0^1 \left(\frac{1}{x+1}\pi - (-1 - 1)\right) dx = [\pi \ln|x + 1| + 2x]_0^1 = \pi(\ln 2 - \ln 1) + 2 = \pi \ln 2 + 2$

(b) $\displaystyle\int_{-2}^0 \int_0^y (x + 2y^2)\, dx\, dy =$

Solution: $int = \int_{-2}^0 \left[\frac{1}{2}x^2 + 2y^2 x\right]_{x=0}^{x=y} dy = \int_{-2}^0 \left(\frac{1}{2}y^2 + 2y^3\right) dy = \left[\frac{1}{6}y^3 + \frac{1}{2}y^4\right]_{-2}^0 = 0 - \left(\frac{1}{6}(-8) + \frac{1}{2}16\right) = \frac{4}{3} - 8 = -6\frac{2}{3}$

(c) $\displaystyle\int\int_R \frac{dy\ dx}{\sqrt{9-x^2-y^2}}$, where R is the *second quadrant* region bounded by $x^2+y^2=4$.

Solution: We use the polar coordinates, in which the region R is given as $0 \le r \le 2$ and $\pi/2 \le \theta \le \pi$. So, in the second equation using substitution $u = 9 - r^2$,

$int = \int_{\pi/2}^{\pi} \int_0^2 (9-r^2)^{-1/2} r\ dr\ d\theta = \int_{\pi/2}^{\pi} \left[-(9-r^2)^{1/2}\right]_0^2\ d\theta =$
$\int_{\pi/2}^{\pi} \left[-\left((9-4)^{1/2} - 9^{1/2}\right)\right]_0^2\ d\theta = \left[3-\sqrt{5}\right]_{\pi/2}^{\pi} = \frac{3-\sqrt{5}}{2}\pi.$

Ex. 3. Find the mass of the solid bounded by the hemisphere $x^2+y^2+z^2 \le R^2$, $z \ge 0$, with the density $\delta(x,y,z) = x^2+y^2+z^2$.

Solution: We use the spherical coordinates. Since the solid, T, is the upper hemisphere, we get
$mass = \int\int\int_T \delta(x,y,z)\ dV = \int\int\int_T (x^2+y^2+z^2)\ dV = \int_0^{\pi/2}\int_0^{2\pi}\int_0^R (\rho^2)\rho^2 \sin\phi\ d\rho\ d\theta\ d\phi =$
$\int_0^{\pi/2}\int_0^{2\pi} \left[\frac{1}{5}\rho^5 \sin\phi\right]_0^R\ d\theta\ d\phi = \int_0^{\pi/2}\int_0^{2\pi} \frac{1}{5}R^5 \sin\phi\ d\theta\ d\phi = \int_0^{\pi/2} \left[\left(\frac{1}{5}R^5\sin\phi\right)\theta\right]_0^{2\pi}\ d\phi =$
$\int_0^{\pi/2} \frac{2}{5}\pi R^5 \sin\phi\ d\phi = \left[\frac{2}{5}\pi R^5(-\cos\phi)\right]_0^{\pi/2} = -\frac{2}{5}\pi R^5(\cos(\pi/2)-\cos 0) = -\frac{2}{5}\pi R^5(0-1) = \frac{2}{5}\pi R^5$

Ex. 4. Find the mass of the plane lamina bounded by $x=0$ and $x = 9-y^2$ with density $\delta(x,y) = y^2$.

Solution: Notice that $x=0$ and $x=9-y^2$ when $9-y^2=0$ that is, when $y=\pm 3$.
$mass = \int\int_R \delta(x,y)dA = \int_{-3}^{3}\int_0^{9-y^2} y^2\ dx\ dy = \int_{-3}^{3} [xy^2]_0^{9-y^2}\ dy = \int_{-3}^{3} y^2(9-y^2)\ dy =$
$\int_{-3}^{3}(9y^2-y^4)\ dy = \left[3y^3 - \frac{1}{5}y^5\right]_{-3}^{3} = 3(27+27) - \frac{1}{5}(243+243) = 162 - 97.2 = 64.8$

Ex. 5. Evaluate $\displaystyle\int_C xy\,ds$, where C is the parametric curve for which $x = 3t$, $y = t^4$, and $0 \le t \le 1$.

Solution: Since $ds = \sqrt{(x'(t))^2 + (y'(t))^2}\ dt = \sqrt{(3)^2+(4t^3)^2}\ dt = \sqrt{9+16t^6}\ dt$,

$\int_C xy\,ds = \int_0^1 (3t)(t^4)\sqrt{9+16t^6}\ dt = \int_0^1 (9+16t^6)^{1/2}\ (3t^5 dt)$
For $u = 9+16t^6$, we get $\frac{du}{dx} = 6\cdot 16t^5$, and so $3t^5dt = \frac{1}{32}\ du$.
Hence, $\int(9+16t^6)^{1/2}\ (3t^5dt) = \int u^{1/2}\frac{1}{32}\ du = \frac{1}{3\cdot 16}u^{3/2} + C = \frac{1}{48}(9+16t^6)^{3/2} + C$. Thus
$\int_C xy\,ds = \left[\frac{1}{48}(9+16t^6)^{3/2}\right]_0^1 = \frac{1}{48}[(9+16)^{3/2} - 9^{3/2}] = \frac{1}{48}[125-27] = \frac{49}{24} = 2\frac{1}{24}$

Ex. 6. Evaluate the integral, where C is the graph of $y = x^3$ from $(-1,-1)$ to $(1,1)$

$\int_C y^2\,dx + xdy =$

Solution: Clearly x changes from -1 to 1. Put $x = t$. Then $y(t) = t^3$ and $-1 \le t \le 1$ and

$\int_C y^2\,dx + xdy = \int_{-1}^{1}(y(t))^2\,x'(t)\,dt + x(t)\,y'(t)\,dt = \int_{-1}^{1}[(t^3)^2\,1 + t\,(3t^2)]\,dt = \int_{-1}^{1}(t^6+3t^3)\,dt =$

$\left[\frac{1}{7}t^7 + \frac{3}{4}t^4\right]_{-1}^{1} = \frac{1}{7}(1+1) + \frac{3}{4}(1-1) = \frac{2}{7}$

Ex. 7. Determine if the following vector field is conservative. Find potential function for a field, if it is conservative.

(a) $\mathbf{F} = \left(x^3 + \frac{y}{x}\right)\mathbf{i} + (y^2 + \ln x)\mathbf{j}$

Solution: We have $P = x^3 + \frac{y}{x}$ and $Q = y^2 + \ln x$. So $\frac{\partial P}{\partial y} = \frac{1}{x}$ and $\frac{\partial Q}{\partial x} = \frac{1}{x}$. Since $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, the field is conservative and we can find the potential function $f(x,y)$. We have

$f(x,y) = \int P\,dx = \int x^3 + \frac{y}{x}\,dx = \frac{1}{4}x^4 + y\ln(x) + K(y)$.

Taking partial derivative, in terms of y, of both side we get

$\ln(x) + K'(y) = \frac{\partial f}{\partial y} = Q = y^2 + \ln x$, so that $K'(y) = y^2$ and $K(y) = \frac{1}{3}y^3 + C$.

Answer: The potential function $f(x,y) = \frac{1}{4}x^4 + y\ln(x) + \frac{1}{3}y^3 + C$.

(b) $\mathbf{F} = (y\cos x + \ln y)\,\mathbf{i} + \left(\frac{x}{y} + e^y\right)\mathbf{j}$

Solution: We have $P = y\cos x + \ln y$ and $Q = \frac{x}{y} + e^y$. So $\frac{\partial P}{\partial y} = \cos x + \frac{1}{y}$ and $\frac{\partial Q}{\partial x} = \frac{1}{y}$. Since $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$, the field is not conservative and the potential function does not exist.

Ex. 8. Find a potential function of the vector field and use the fundamental theorem for line integrals to evaluate

$\int_{(\pi/2,\pi/2)}^{(\pi,\pi)} (\sin y + y\cos x)\,dx + (\sin x + x\cos y)\,dy =$

Solution: We have $P = \sin y + y\cos x$ and $Q = \sin x + x\cos y$. It is easy to see that $\frac{\partial P}{\partial y} = \cos y + \cos x = \frac{\partial Q}{\partial x}$ so indeed we can find the potential function $f(x,y)$. We have

$f(x,y) = \int P\,dx = \int \sin y + y\cos x\,dx = x\sin y + y\sin x + K(y)$.

Taking partial derivative, in terms of y, of both side we get

$x\cos y + \sin x + K'(y) = \frac{\partial f}{\partial y} = Q = \sin x + x\cos y$, so that $K'(y) = 0$ and $K(y) = C$.

So, the potential function $f(x,y) = x\sin y + y\sin x + C$ and

$int = [f(x,y)]_{(\pi/2,\pi/2)}^{(\pi,\pi)} = [x\sin y + y\sin x]_{(\pi/2,\pi/2)}^{(\pi,\pi)} = (\pi\sin\pi + \pi\sin\pi) - (\frac{\pi}{2}\sin\frac{\pi}{2} + \frac{\pi}{2}\sin\frac{\pi}{2}) =$
$(0+0) - (\frac{\pi}{2} + \frac{\pi}{2}) = -\pi$

Ex. 9. Apply Green's theorem to evaluate the following integral, where the simple closed curve C, with counter clockwise direction, is the boundary of the circle $x^2 + y^2 = 1$.

$\oint_C (\sin x - x^2 y)\, dx + xy^2\, dy =$

Solution: Let D denoted the disk $x^2 + y^2 \leq 1$.

By Green's theorem $int = \int \int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)\ dA$, where $P = \sin x - x^2 y$ and $Q = xy^2$. So,

$int = \int \int_D \left(y^2 - (-x^2) \right)\ dA = \int \int_D \left(x^2 + y^2 \right)\ dA$

Changing to the polar coordinates, we get

$int = \int_0^{2\pi} \int_0^1 r^2\ r\ dr\ d\theta = \int_0^{2\pi} \left[\frac{1}{4} r^4 \right]_0^1\ d\theta = \int_0^{2\pi} \frac{1}{4}\ d\theta = \left[\frac{1}{4} \theta \right]_0^{2\pi} = \frac{1}{2} \pi$