MATH 251
Instr. K. Ciesielski
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SAMPLE TEST # 3

Solve the following exercises. **Show your work.** (No credit will be given for an answer with no supporting work shown.)

Ex. 1. Show that the following limit does not exist

$$\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^4+7y^4}$$

Solution:

On x-axis, $y=0$: $L_1 = \lim_{x\to 0} \frac{x^3\cdot 0}{x^4+0} = 0$.

On the line $y=x$: $L_2 = \lim_{x\to 0} \frac{x^3\cdot x}{x^4+7x^4} = \lim_{x\to 0} \frac{x^4}{8x^4} = \frac{1}{8}$.

Answer: Limit does not exist as $L_1 \neq L_2$.

Ex. 2. Compute the first order partial derivatives of $f(x,y,z) = ze^{x^2}\cos y$.

Solution:

$\frac{\partial f}{\partial x} = f_x = ze^{x^2}\cos y \cdot 2x = 2xze^{x^2}\cos y$

$\frac{\partial f}{\partial y} = f_y = ze^{x^2}(-\sin y) = -ze^{x^2}\sin y$

$\frac{\partial f}{\partial z} = f_z = e^{x^2}\cos y$

Ex. 3. Compute all second order partial derivatives of $g(s,t) = e^{5t} + t\sin(3s)$.

Solution:

$g_s = 3t\cos(3s)$ $\quad$ $g_{ss} = -9t\sin(3s)$ $\quad$ $g_{st} = 3\cos(3s)$

$g_t = 5e^{5t} + \sin(3s)$ $\quad$ $g_{ts} = 3\cos(3s)$ $\quad$ $g_{tt} = 25e^{5t}$

Ex. 4. Find an equation of the plane tangent to the surface $z = x^2 - 5y^3$ at the point $P(2,1,-1)$.

Solution:

$z_x = 2x$; $\quad z_x(P) = 2\cdot 2 = 4$;

$z_y = -15y^2$; $\quad z_y(P) = -15\cdot 1^2 = -15$;

Normal vector $\mathbf{n} = \langle z_x(P), z_y(P), -1\rangle = \langle 4,-15,-1\rangle$.

Answer: $4(x-2) - 15(y-1) - 1(z+1) = 0$ or $4x - 15y - z + 6 = 0$.

Ex. 5. Find the absolute maximum and the absolute minimum of the function $f(x,y) = x^3 - xy$ on the region bounded below by parabola $y = x^2 - 1$ and above by line $y = 3$. You will get credit **only** if **all** critical points are found.

Solution: The curves intersect, when $x^2 - 1 = 3$, that is, when $x = \pm 2$.
Thus, we need to consider the region above $x^2 - 1$ and below 3 for x in the interval $[-2, 2]$.

Region's interior: $f_x(x,y) = 3x^2 - y$ and $f_y(x,y) = -x$. This leads to system $3x^2 - y = 0$ and $-x = 0$, with only solution $(x,y) = (0,0)$. This point belongs to the region. This is our first critical point.

Lower boundary: $y = x^2 - 1$ **and** $-2 \le x \le 2$**.** Then
$g(x) = f(x, x^2 - 1) = x^3 - x(x^2 - 1) = x$ and $g'(x) = 1$ is never 0.
So, there are no true critical points, but we need to consider the endpoints of g, $x = \pm 2$.
This give us the critical points $(x,y) = (\pm 2, 3)$.

Upper boundary: $y = 3$ **and** $-2 \le x \le 2$**.** Then
$g(x) = f(x, 3) = x^3 - 3x$ and $g'(x) = 3x^2 - 3$, which is 0 when $x = \pm 1 \in [-2, 2]$.
This give us the critical points $(x,y) = (\pm 1, 3)$. (Plus the end points $(x,y) = (\pm 2, 3)$, considered above.)

Checking the critical points: $f(0,0) = 0$;
$f(2,3) = 2^3 - 6 = 2$; $f(-2,3) = (-2)^3 + 6 = -2$;
$f(1,3) = 1^3 - 3 = -2$; $f(-1,3) = (-1)^3 + 3 = 2$;

Answer: f has the absolute maximum value 2, at points $(2,3)$ and $(-1,3)$.
f has the absolute minimum value -2, at points $(-2,3)$ and $(1,3)$.

Ex. 6. Find the volume of the solid bounded above by the surface $z = 28xy$, bounded below by xy-plane, and which is above the region bounded by $y = x^6$ and $y = x$.

Solution: The curves intersect, when $x^6 = x$, that is, when $x = 0$ and $x = 1$.
Thus, we need to find an integral above x^6 and below x, on the interval $[0, 1]$:

$$\int_0^1 \int_{x^6}^{x} 28xy\ dy\ dx = \int_0^1 \left[14xy^2\right]_{y=x^6}^{y=x}\ dx = \int_0^1 \left[14x^3 - 14x^{13}\right]_{y=x^6}^{y=x}\ dx = \left[\tfrac{14}{4}x^4 - x^{14}\right]_{x=0}^{x=1} = 2.5$$

Ex. 7. Evaluate $\int_0^1 \int_0^x 4e^{x^2}\ dy\ dx$

Solution: $\int_0^1 \int_0^x 4e^{x^2}\ dy\ dx = \int_0^1 \left[4e^{x^2}y\right]_{y=0}^{y=x}\ dx = \int_0^1 4(e^{x^2}x - e^{x^2}0)\ dx = \int_0^1 4e^{x^2}x\ dx$

Using substitution $v = x^2$, we obtain that it is equal

$\left[2e^{x^2}\right]_{x=0}^{x=1} = 2(e^1 - e^0) = 2(e - 1)$.

Ex. 8. Find the point on the cone $z = \sqrt{x^2 + y^2}$ which is the closest to the point $(4, -8, 0)$.

Solution: Distance of (x, y, z) on the surface from $(4, -8, 0)$ is $\sqrt{(x-4)^2 + (y+8)^2 + (z-0)^2}$.
Since $z^2 = x^2 + y^2$, this is equal to

$f(x, y) = \sqrt{(x-4)^2 + (y+8)^2 + (x^2 + y^2)}$.

$f_x(x, y) = \frac{2(x-4)+2x}{2\sqrt{(x-4)^2+(y+8)^2+(x^2+y^2)}}$ and $f_y(x, y) = \frac{2(y+8)+2y}{2\sqrt{(x-4)^2+(y+8)^2+(x^2+y^2)}}$.

$f_x = 0$ when $2(x-4) + 2x = 0$, that is, $4x - 8 = 0$, so $x = 2$.

$f_y = 0$ when $2(y+8) + 2y$, that is, $4y + 16 = 0$, so $y = -4$.

This gives critical point $(2, -4)$. Since these are the coordinates of a point on the cone, we get $z = \sqrt{2^2 + (-4)^2} = \sqrt{20}$.

Answer: Point $(2, -4, \sqrt{20})$.

Ex. 9. Find the directional derivative of $f(x, y) = 10e^y \sin x$ at the point $P(\pi/4, 0)$ in the direction of the vector $\mathbf{v} = 4\mathbf{i} - 3\mathbf{j}$.

Solution: The unit vector in the direction of $\mathbf{v}$ is equal

$\mathbf{u} = \frac{1}{|\mathbf{v}|}\mathbf{v} = \frac{1}{\sqrt{4^2+(-3)^2}}\mathbf{v} = \frac{1}{5}\langle 4, -3\rangle = \langle .8, -.6\rangle$.

$f_x(x, y) = 10e^y \cos x$; $f_x(P) = 10e^0 \cos(\pi/4) = 5\sqrt{2}$.

$f_y(x, y) = 10e^y \sin x$; $f_x(P) = 10e^0 \sin(\pi/4) = 5\sqrt{2}$.

$\nabla f(P) = \langle f_x(P), f_y(P)\rangle = \langle\sqrt{2}/2, \sqrt{2}/2\rangle$.

$D_{\mathbf{u}}f(P) = \nabla f(P) \cdot \mathbf{u} = \langle 5\sqrt{2}, 5\sqrt{2}\rangle \cdot \langle .8, -.6\rangle = (5\sqrt{2})(.8) + (5\sqrt{2})(-.6) = \sqrt{2}$.

Ex. 10. Find the gradient of $g(x, y, z) = x^2 + e^{yz} + \cos(x + 2y)$.

Solution: $\nabla g(x, y, z) = \langle g_x, g_y, g_z\rangle = \langle 2x - \sin(x+2y), ze^{yz} - 2\sin(x+2y), ye^{yz}\rangle$.